1. Let $f_n(x) = \frac{x^n(1-x^2)}{\sqrt{1+x}}$ for $0 \leq x \leq \frac{1}{2}$. Since

$$|f_n(x)| = \frac{|x^n|1-x^2}{\sqrt{1+x}} \leq \frac{(\frac{1}{2})^n \cdot \frac{1}{2}}{1} = \left(\frac{1}{2}\right)^n$$

for $0 \leq x \leq \frac{1}{2}$ and the series $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent by the geometric series, the series of functions $\sum_{n=0}^{\infty} \frac{x^n(1-x^2)}{\sqrt{1+x}}$ converges uniformly on $[0, \frac{1}{2}]$ by the $M$-test. Note that each $f_n(x)$ is Riemann integrable on $[0, \frac{1}{2}]$. We have

$$\sum_{n=0}^{\infty} \int_{0}^{\frac{1}{2}} \frac{x^n(1-x^2)}{\sqrt{1+x}} \, dx = \int_{0}^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{x^n(1-x^2)}{\sqrt{1+x}} \, dx = \int_{0}^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} x^n \right) \cdot \frac{1-x^2}{\sqrt{1+x}} \, dx$$

$$= \int_{0}^{\frac{1}{2}} \frac{1}{1-x} \cdot \frac{1-x^2}{\sqrt{1+x}} \, dx = \int_{0}^{\frac{1}{2}} \sqrt{1+x} \, dx = \frac{2}{3} (1+x)^\frac{3}{2} \bigg|_{0}^{\frac{1}{2}} = \frac{2}{3} \left[ \left(\frac{3}{2}\right)^\frac{3}{2} - 1 \right] = \sqrt{\frac{3}{2}} - \frac{2}{3}.$$
for $0 \leq x \leq 1$ and the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent by the ratio test, the series of functions $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}$ converges uniformly to $e^{-x^3}$ on $[0, 1]$ by the $M$-test. Note that each $\frac{(-1)^n x^{3n}}{n!}$ is Riemann integrable. Thus

$$
\int_0^1 e^{-x^3} \, dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} \, dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{3n}}{n!} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(3n + 1)}.
$$

Let $a_n = \frac{1}{n!(3n + 1)}$. Then the sequence $\{a_n\}$ is positive, monotone decreasing and $\lim_{n \to \infty} a_n = 0$. By applying the alternating test estimation, from $a_{n+1} < 0.001$ or $(n + 1)!(3n + 4) \geq 1000$, we have $n \geq 4$ and so

$$
\int_0^1 e^{-x^3} \, dx \approx 1 - \frac{1}{4!} \cdot 4 + \frac{1}{2!} \cdot 7 - \frac{1}{3!} \cdot 10 + \frac{1}{4!} \cdot 13 = 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312}
$$

with error less than 0.001.

4. Let $f_n(x) = \frac{\cos^n x}{n^3}$. Given any point $x_0$ in $(\infty, +\infty)$, let $a$ and $b$ be any numbers such that $a < x_0 < b$. Then

(1) Each $f_n'(x) = \frac{-\cos^{n-1} x \sin x}{n^2}$ is continuous on $[a, b]$.

(2) The series of functions $\sum_{n=1}^{\infty} f_n(x)$ absolutely converges on $[a, b]$ by the comparison test because $\left| \frac{\cos^n x}{n^3} \right| \leq \frac{1}{n^3}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by the $p$-series. So it converges pointwise on $[a, b]$.

(3) The series of functions $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[a, b]$ by the $M$-test because $\left| \frac{-\cos^{n-1} x \sin x}{n^2} \right| \leq \frac{1}{n^2}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the $p$-series.

Thus the function $f(x) = \sum_{n=1}^{\infty} \frac{\cos^n x}{n^3}$ is differentiable on $[a, b]$ and so at $x_0$. Since $x_0$ is any given point in $(\infty, +\infty)$, the function $f(x)$ is differentiable on $(\infty, +\infty)$.

5. Let $f_n(x) = \frac{x^{2n}}{(2n)!}$. Given any point $x_0 \in (\infty, +\infty)$, let $a$ and $b$ be numbers such that $a < x_0 < b$. Note that

(1) $f_0'(x) = 0$ and $f_n'(x) = \frac{x^{2n-1}}{(2n-1)!}$ for $n \geq 1$ are continuous on $[a, b]$. 
(2). The series of functions \( \sum_{n=0}^{\infty} f_n(x) \) absolutely converges on \([a, b]\) by the ratio test. So it converges pointwise on \([a, b]\).

(3). The series of functions \( \sum_{n=0}^{\infty} f'_n(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} \) converges uniformly on \([a, b]\) by the \(M\)-test because
\[
\left| \frac{x^{2n-1}}{(2n-1)!} \right| \leq \frac{\max\{|a|, |b|\}^{2n-1}}{(2n-1)!}
\]
and the series \( \sum_{n=1}^{\infty} \frac{\max\{|a|, |b|\}^{2n-1}}{(2n-1)!} \) converges by the ratio test.

Thus the function \( f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \) is differentiable on \([a, b]\) and so at \(x_0\). Thus \( f(x) \) is differentiable on \((-\infty, +\infty)\) with
\[
f'(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}
\]
for \(x \in (-\infty, +\infty)\).

Similarly, the function \( f'(x) \) is differentiable with
\[
f''(x) = \sum_{n=1}^{\infty} \left( \frac{x^{2n-1}}{(2n-1)!} \right)' = \sum_{n=1}^{\infty} \frac{x^{2n-2}}{(2n-2)!} = f(x)
\]
and so \( y = f(x) \) is a solution to \( y'' = y \).

6. Let \( S(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) and \( S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \). Then, for \( x > 0 \), we have
\[
|S_n(x) - S(x)| = \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \geq \frac{x^{n+1}}{(n+1)!}
\]
because each \( \frac{x^k}{k!} \geq 0 \). Thus
\[
T_n = \sup_{-\infty < x < \infty} |S_n(x) - S(x)| \geq \sup_{0 < x < +\infty} |S_n(x) - S(x)| \geq \sup_{0 < x < +\infty} \frac{x^{n+1}}{(n+1)!} = +\infty
\]
because \( x^{n+1} \to +\infty \) as \( x \to \infty \). Since \( T_n \) does not tend to 0, the sequence of the functions \( S_n(x) \) does not converge uniformly to \( S(x) \) on \((-\infty, +\infty)\) and so the series of functions \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) does not converge uniformly on \((-\infty, +\infty)\).