More Examples to Section 3.2

Let \( \{F_n\} \) be a sequence of functions on an interval \( I \). To see whether \( \{F_n\} \) is uniformly convergent, we may do by the following steps.

1. Determine the limiting function \( F(x) = \lim_{n \to \infty} F_n(x) \).
2. Determine \( T_n = \sup_{x \in I} |F_n(x) - F(x)| \).
3. Check whether \( \lim_{n \to \infty} T_n = 0 \).

**Note.** The main point is to see whether \( \lim_{n \to \infty} T_n = 0 \) or not.

**Example 1.** Determine whether the following sequences of functions converge uniformly on the indicated interval.

(a) \( f_n(x) = \frac{n^2 \ln x}{x^n}, \ x \in [1, +\infty) \);

(b) \( f_n(x) = \frac{n^2 \ln x}{x^n}, \ x \in [2, +\infty) \).

**Solution.** Let \( f(x) = \lim_{n \to \infty} f_n(x) = 0 \) for \( x \geq 1 \).

(a). \( T_n = \sup_{x \geq 1} |f_n(x) - 0| = \sup_{x \geq 1} \frac{n^2 \ln x}{x^n} = \sup f_n(x) \).

From

\[
    f_n'(x) = n^2 \left( \frac{1}{x} \cdot x^{-n} \right) - n^3 \ln x \cdot x^{-n-1} = \frac{n^2 - n^3 \ln x}{x^{n+1}} = 0,
\]

we have \( n^2 - n^3 \ln x = 0 \) or \( x = e^{\frac{1}{n}} \). Observe that \( f_n(x) \) is monotone increasing for \( 1 \leq x \leq e^{\frac{1}{n}} \) and monotone decreasing for \( x \geq e^{\frac{1}{n}} \). Thus

\[
    T_n = \max_{x \geq 1} f_n(x) = f_n(e^{\frac{1}{n}}) = \frac{n^2 \cdot \frac{1}{n}}{(e^{\frac{1}{n}})^n} = \frac{n}{e} \not\to 0
\]

as \( n \to \infty \) and so \( \{f_n(x)\} \) does NOT converge uniformly.

(b). Since \( e^{\frac{1}{n}} \leq 2 \) for \( n \geq 2 \), the function \( f_n(x) \) is monotone decreasing on \( [2, +\infty) \) for \( n \geq 2 \) and so \( T_n = \sup_{x \geq 2} |f_n(x) - f(x)| = f_n(2) = \frac{1}{e} \not\to 0 \) as \( n \to \infty \).
\[ \frac{n^2 \ln 2}{2^n} \] for \( n \geq 2 \). Since \( \lim_{n \to \infty} T_n = 0 \), \( \{f_n\} \) converges uniformly on \([2, +\infty)\). □

Sometimes we do not need to figure out \( T_n \) EXPLICITLY.

**Example 2.** Show that \( F_n(x) = \frac{n^2 \ln x \sin nx}{x^n} \) converges uniformly on \([2, +\infty)\).

**Solution.** \( F(x) = \lim_{n \to \infty} F_n(x) = 0 \) for \( x \geq 2 \). Observe

\[
T_n = \sup_{x \geq 2} |F_n(x) - F(x)| = \sup_{x \geq 2} \frac{n^2 \ln x |\sin nx|}{x^n} \leq \frac{n^2 \ln 2}{2^n}
\]

for \( n \geq 2 \). Since \( \lim_{n \to \infty} T_n = 0 \), \( \{F_n\} \) converges uniformly. □