Question 1. Answer: sup $S = 1$ and inf $S = 0$. We show that sup $S = 1$. Let $r$ be any element in $S$. By the definition, $r$ is rational number with $0 \leq r < 1$. Thus 1 is an upper bound of $S$ and 0 is a lower bound of $S$. Let $M$ be any upper bound of $S$. Then $r \leq M$ for any rational number $r$ with $0 \leq r < 1$. In particular, $\frac{n}{n+1} \leq M$ for any positive integer $n$. It follows that $1 = \lim_{n \to \infty} \frac{n}{n+1} \leq M$ and so 1 is the least upper bound of $S$ or sup $S = 1$. Let $m$ be any lower bound of $S$. Then $m \leq r$ for any rational number $r$ with $0 \leq r < 1$. In particular, $m \leq 0$ and so 0 is the greatest lower bound of $S$ or inf $S = 0$. □

Question 2. Since inf $B$ is a lower bound of $B$, we have inf $B \leq x$ for any $x \in B$ and so inf $B \leq y$ for any $y \in A \subseteq B$. It follows that inf $B$ is a lower bound of $A$. Thus inf $B \leq$ inf $A$ because inf $A$ is the greatest lower bound of $A$. □

Question 3 (i). First we show that max{sup $A$, sup $B$} is an upper bound of $A \cup B$. Let $z$ be any element in $A \cup B$. Then $z \in A$ or $B$. If $z \in A$, then $z \leq$ sup $A \leq$ max{sup $A$, sup $B$}. Otherwise, $z \in B$ and $z \leq$ sup $B \leq$ max{sup $A$, sup $B$}. Thus max{sup $A$, sup $B$} is an upper bound of $A \cup B$.

Now we show that max{sup $A$, sup $B$} is the least upper bound of $A \cup B$. Let $M$ be any upper bound of $A \cup B$. Then $z \leq M$ for any $z \in A \cup B$. In particular, $z \leq M$ for $z \in A \subseteq A \cup B$ and so $M$ is an upper bound of $A$. It follows that sup $A \leq M$. Similarly, we have sup $B \leq M$. Thus max{sup $A$, sup $B$} $\leq$ $M$ and so max{sup $A$, sup $B$} = sup $A \cup B$. □

Question 3 (ii). No, it is not true. An counter-example is as follows. Let $A = \{1, 2\}$ and let $B = \{1, 3\}$. Then sup $A =$ max $A = 2$ and sup $B =$ max $B = 3$. It follows that min{sup $A$, sup $B$} = min{$2, 3$} $= 2$. But sup $A \cap B =$ sup{$1$} $= 1 \neq$ min{sup $A$, sup $B$}. □
Question 4 (i). We prove that $2 \leq a_n \leq 3$ by induction on $n$. Since $a_1 = 2$, we have $2 \leq a_1 \leq 3$. Suppose that $a_{n-1} \leq 3$ with $n \geq 2$. Then
\[ 2 \leq \sqrt{6 + 2} \leq a_n = \sqrt{6 + a_{n-1}} \leq \sqrt{6 + 3} = 3. \]
The induction is finished and hence the statement.

Question 4 (ii). Let $n \geq 2$. Then
\[ a_n - a_{n-1} = \sqrt{6 + a_{n-1}} - a_{n-1} = \frac{(\sqrt{6 + a_{n-1}} - a_{n-1})(\sqrt{6 + a_{n-1}} + a_{n-1})}{\sqrt{6 + a_{n-1}} + a_{n-1}} \]
\[ = \frac{6 + a_{n-1} - a_{n-1}^2}{\sqrt{6 + a_{n-1}} + a_{n-1}} \geq 0 \]
because $\sqrt{6 + a_{n-1}} + a_{n-1} > 0$ and $6 + x - x^2 = -(x-3)(x+2) \geq 0$ for $-2 \leq x \leq 3$. Thus $\{a_n\}$ is monotone increasing.

Question 4 (iii). By (i) and (ii), $\{a_n\}$ is bounded above and monotone increasing. Thus $\{a_n\}$ is convergent. Let $A = \lim_{n \to \infty} a_n$. Then we have the equation
\[ A = \lim_{n \to \infty} a_n = \lim_{n \to \infty} \sqrt{6 + a_{n-1}} = \sqrt{6 + \lim_{n \to \infty} a_{n-1}} = \sqrt{6 + A} \]
and so $A^2 = 6 + A$. It follows that $A = -2$ or 3. Since $a_n \geq 2$ for each $n$, $A = \lim_{n \to \infty} a_n \geq 2$ and so $A = 3$.

Question 5. First we show that $0 \leq x_n \leq 1$ by induction on $n$. When $n = 1$, we have $0 \leq x_1 = \frac{3}{4} \leq 1$. Suppose that $0 \leq x_{n-1} \leq 1$ with $n \geq 2$. Observe that
\[ x_n = 2x_{n-1} - x_{n-1}^2 = 1 - (1 - 2x_{n-1} + x_{n-1}^2) = 1 - (1 - x_{n-1})^2. \]
Since $0 \leq 1 - x_{n-1} \leq 1$ by induction, we have $0 \leq x_n \leq 1$. The induction is finished and so $0 \leq x_n \leq 1$ for all $n$.

Observe that
\[ x_{n+1} - x_n = (2x_n - x_n^2) - x_n = x_n - x_n^2 = x_n(1 - x_n). \]
Since $0 \leq x_n \leq 1$, we have $x_{n+1} - x_n = x_n(1 - x_n) \geq 0$ and so the sequence $\{x_n\}$ is monotone increasing and bounded. Thus the limit of $\{x_n\}$ exists. Let $A = \lim_{n \to \infty} x_n$.

From the equation $x_{n+1} = 2x_n - x_n^2$, we have
\[ A = \lim_{n \to \infty} x_{n+1} = 2 \lim_{n \to \infty} x_n - \left( \lim_{n \to \infty} x_n \right)^2 = 2A - A^2. \]
and so $A = 0$ or 1. Since $x_n \geq x_1 = \frac{3}{4}$, we have $A = \lim_{n \to \infty} x_n \geq \frac{3}{4}$ and so $A \neq 0$. Thus $A = 1$. □

Question 6 (a). Since $\{a_n\} = \{4 + \cos \frac{n\pi}{2}\} = \{4, 3, 4, 5, 4, 3, 4, 5, 4, 3, 4, 5, \ldots\}$, we have

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} = 5 \quad \text{and} \quad c_n = \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\} = 3$$

for all $n$ and so $\lim_{n \to \infty} 4 + \cos \frac{n\pi}{2} = \lim_{n \to \infty} b_n = 5$ and $\lim_{n \to \infty} 4 + \cos \frac{n\pi}{2} = \lim_{n \to \infty} c_n = 3$. □

Question 6 (b). Observe that

$$0 \leq \frac{1 + (-1)^n}{n} \leq \frac{2}{n}.$$ 

Since $\lim_{n \to \infty} \frac{2}{n} = \lim_{n \to \infty} 0 = 0$, we have $\lim_{n \to \infty} \frac{1 + (-1)^n}{n} = 0$ and so

$$\lim_{n \to \infty} \frac{1 + (-1)^n}{n} = \lim_{n \to \infty} \frac{1 + (-1)^n}{n} = \lim_{n \to \infty} \frac{1 + (-1)^n}{n} = 0.$$ □