2002/2003 Semester I  MA2108  Advanced Calculus II

Solutions to Tutorial 4

Question 1 (a). Let \( a_n = \frac{n^2 - 1}{2n^2 + n} \). Then \( \lim_{n \to \infty} a_n = \frac{1}{2} \neq 0 \) and so the series \( \sum_{n=1}^{\infty} \frac{n^2 - 1}{2n^2 + n} \) is divergent by the divergence test. \( \square \)

Question 1 (b). Let \( a_n = \sin \frac{n\pi}{2} \). Then \( \{a_n\} = \{1, 0, -1, 0, 1, 0, -1, 0, \ldots\} \) and so \( \lim_{n \to \infty} a_n \) does not exist. Thus the series \( \sum_{n=1}^{\infty} \sin \frac{n\pi}{2} \) is divergent by the divergence test. \( \square \)

Question 1 (c). Let \( a_n = \frac{n^2 + 1 + \ln n}{n + n^3 + 4} \) and let \( b_n = \frac{1}{n} \). Then
\[
\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{n + n^3 + 4}{n^2 + 1 + \ln n} = \lim_{n \to \infty} \frac{1}{n^2} + \frac{1}{n} + \frac{4}{n^3} = 0 + 1 + 0 = 1.
\]
Since \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent, so is \( \sum_{n=1}^{\infty} \frac{n^2 + 1 + \ln n}{n + n^3 + 4} \) by the limit comparison test. \( \square \)

Question 1 (d). Observe that
\[
\frac{3 + \sin n}{n^2} \leq \frac{4}{n^2}.
\]
Since \( \sum_{n=1}^{\infty} \frac{4}{n^2} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent by the \( p \)-series, the positive series \( \sum_{n=1}^{\infty} \frac{3 + \sin n}{n^2} \) is convergent by the comparison test. \( \square \)

Question 1 (e). Observe that
\[
\frac{2^n + 3}{3^{n+1} - n} \leq \frac{2^n + 2^n}{3^{n+1} - n} = \frac{2^{n+1}}{3^{n+1}} = \left(\frac{2}{3}\right)^{n+1}
\]
for \( n \geq 2 \). Since \( \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n+1} \) is convergent by the geometric series, the positive series \( \sum_{n=1}^{\infty} \frac{2^n + 3}{3^{n+1} - n} \) is convergent by the comparison test. \( \square \)

Question 1 (f). Let \( a_n = \frac{2}{n^{1+\frac{1}{2}}} \) and let \( b_n = \frac{1}{n} \). Observe that
\[
\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{n^{1+\frac{1}{2}}}{2} = \lim_{n \to \infty} \frac{\sqrt{n}}{2} = \frac{1}{2}.
\]
Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{2}{n^{1+\pi}} \) is divergent by the limit comparison test.

**Question 1 (g).** Observe that
\[
4 + (-1)^n \geq \frac{3}{2n}.
\]

Since \( \sum_{n=1}^{\infty} \frac{3}{2n} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{4 + (-1)^n}{2n} \) is divergent by the comparison test.

**Question 1 (h).** Observe that
\[
\frac{1}{n(1 + \ln n)^p} = \frac{(1 + \ln n)^{-p}}{n} \geq \frac{1}{n}
\]
for \( p \leq 0 \). Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)^p} \) is divergent for \( p \leq 0 \) by the comparison test.

**Question 1 (i).** Observe that
\[
\frac{n}{n^2 + 1} \geq \frac{n}{n^2 + n^2} = \frac{n}{2n^2} = \frac{1}{2n}.
\]

Since \( \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \) is divergent by the comparison test.

**Question 2 (a).** Since \( \sum_{n=1}^{\infty} a_n \) is convergent, we have \( \lim_{n \to \infty} a_n = 0 \) and so there exists a positive integer \( N \) such that \( a_n = |a_n| = |a_n - 0| < 1 \) for \( n > N \). It follows
\[
a_n^2 = a_n \cdot a_n \leq 1 \cdot a_n = a_n
\]
for \( n > N \). By the comparison test, the positive series \( \sum_{n=1}^{\infty} a_n^2 \) is convergent.

**Question 2 (b).** Let \( a_n = \frac{1}{n^2} \). Then \( \sum_{n=1}^{\infty} a_n \) is convergent but \( \sum_{n=1}^{\infty} \sqrt{a_n} \) is divergent by the \( p \)-series.
**Question 3 (a).** Let \( f(x) = \frac{1}{x(1 + \ln x)} \). Then \( f(x) \) is a positive monotone decreasing function over \([1, +\infty)\). Since

\[
\int_1^\infty f(x) \, dx = \int_1^\infty \frac{1}{x(1 + \ln x)} \, dx = \int_0^\infty \frac{1}{1 + y} \, dy = \ln(1 + y) \bigg|_0^\infty = +\infty.
\]

is divergent, the series \( \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)} \) is divergent by the integral test. \( \square \)

**Question 3 (b).** Let \( f(x) = \frac{1}{x[1 + (\ln x)^2]} \). Then \( f(x) \) is a positive monotone decreasing function over \([1, +\infty)\). Since

\[
\int_1^\infty f(x) \, dx = \int_1^\infty \frac{1}{x[1 + (\ln x)^2]} \, dx = \int_0^\infty \frac{1}{1 + y^2} \, dy = \arctan y \bigg|_0^\infty = \frac{\pi}{2}
\]

is convergent, the series \( \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)} \) is convergent by the integral test. \( \square \)