Question 1 (a). The limiting function \( F(x) = \lim_{n \to \infty} \frac{n^2}{n^2 + x^2} = \lim_{n \to \infty} \frac{1}{1 + \frac{x^2}{n^2}} = 1. \)

\[
0 \leq T_n = \sup_{0 \leq x \leq 1} |F_n(x) - F(x)| = \sup_{0 \leq x \leq 1} \left| \frac{n^2}{n^2 + x^2} - 1 \right| = \sup_{0 \leq x \leq 1} \frac{x^2}{n^2 + x^2} \leq \frac{1}{n^2}.
\]

Since \( \lim_{n \to \infty} \frac{1}{n^2} = \lim_{n \to \infty} 0 = 0, \) we have \( \lim_{n \to \infty} T_n = 0 \) by the Squeeze theorem and so \( \{F_n\} \) converges uniformly on \([0, 1]\). \( \square \)

Question 1 (b). The limiting function \( F(x) = \lim_{n \to \infty} x^n(1 - x) = 0 \) for \( 0 \leq x \leq 1. \) Observe

\[
T_n = \sup_{0 \leq x \leq 1} |F_n(x) - F(x)| = \sup_{0 \leq x \leq 1} x^n(1 - x).
\]

Let \( g(x) = x^n(1 - x). \) Then \( g'(x) = nx^{n-1}(1 - x) - x^n = x^{n-1}[n - (n + 1)x]. \) From \( g'(x) = 0, \) we have \( x = 0 \) or \( \frac{n}{n + 1}. \) Since \( g'(x) \geq 0 \) for \( 0 \leq x \leq \frac{n}{n + 1} \) and \( g'(x) \leq 0 \) for \( \frac{n}{n + 1} \leq x \leq 1, \) \( \sup_{0 \leq x \leq 1} \) \( g(x) = \max\{g(x)|0 \leq x \leq 1\} = \left( \frac{n}{n + 1} \right)^n \left( 1 - \frac{n}{n + 1} \right) \) and so

\[
\lim_{n \to \infty} T_n = \lim_{n \to \infty} \left( \frac{n}{n + 1} \right)^n \left( 1 - \frac{n}{n + 1} \right) = \lim_{n \to \infty} \frac{1}{(1 + \frac{1}{n})^n} \cdot \frac{1}{n + 1} = \frac{1}{e} \cdot 0 = 0.
\]

Thus \( \{F_n\} \) converges uniformly on \([0, 1]. \) \( \square \)

Question 1 (c). The limiting function \( f(x) = \lim_{n \to \infty} \frac{n \ln x}{x^n} = 0 \) for \( 1 \leq x < \infty. \) Observe that

\[
T_n = \sup_{x \geq 1} |f_n(x) - f(x)| = \sup_{x \geq 1} \left| \frac{n \ln x}{x^n} - 0 \right| = \sup_{x \geq 1} \frac{n \ln x}{x^n}.
\]

Let \( g(x) = \frac{n \ln x}{x^n}. \) From

\[
g'(x) = (n \ln x \cdot x^{-n})' = \frac{1}{x} - \frac{n}{x^n} - n^2 \ln x \cdot x^{-n-1} = \frac{n - n^2 \ln x}{x^{n+1}} = 0,
\]

we have \( x = e^{\frac{1}{n}}. \) Since \( g'(x) \geq 0 \) for \( 1 \leq x \leq e^{\frac{1}{n}} \) and \( g'(x) \leq 0 \) for \( x \geq e^{\frac{1}{n}}, \) we have

\[
T_n = \sup_{x \geq 1} g(x) = \max\{g(x)|x \geq 1\} = \frac{n \ln e^{\frac{1}{n}}}{(e^{\frac{1}{n}})^n} = \frac{1}{e}
\]

and so \( \lim_{n \to \infty} T_n = \frac{1}{e} \neq 0. \) Thus \( \{f_n\} \) does not converge uniformly on \([1, +\infty). \) \( \square \)
Question 1 (d). The limiting function $f(x) = \lim_{n \to \infty} \frac{n \ln x \cos nx}{x^n} = 0$ for $x \geq 4$. Observe that

$$0 \leq T_n = \sup_{x \geq 4} |f_n(x) - f(x)| = \sup_{x \geq 4} \frac{n \ln x \cdot |\cos nx|}{x^n} \leq \sup_{x \geq 4} \frac{n \ln x}{x^n} = g(4) = \frac{n \ln 4}{4^n},$$

where $g(x) = \frac{n \ln x}{x^n}$ is monotone decreasing on $[4, +\infty)$. Since $\lim_{n \to \infty} \frac{n \ln 4}{4^n} = 0$, we have $\lim_{n \to \infty} T_n = 0$ by the Squeeze theorem and so $\{f_n\}$ converges uniformly on $[4, +\infty)$.

Question 1 (e). The limiting function $F(x) = \lim_{n \to \infty} F_n(x) = \lim_{n \to \infty} \frac{n^2}{n^2 + x^2} = 1$. Observe that

$$T_n = \sup_{x \geq 0} |F_n(x) - F(x)| = \sup_{x \geq 0} \left| \frac{n^2}{n^2 + x^2} - 1 \right| = \sup_{x \geq 0} \frac{x^2}{n^2 + x^2}.$$

Let $g(x) = \frac{x^2}{n^2 + x^2}$. Since

$$g'(x) = \frac{2x(n^2 + x^2) - x^2 \cdot 2x}{(n^2 + x^2)^2} = \frac{2xn^2}{(n^2 + x^2)^2} \geq 0$$

for $x \geq 0$, the function $g(x)$ is monotone increasing and so

$$T_n = \sup_{x \geq 0} g(x) = \lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{x^2}{n^2 + x^2} = \lim_{x \to \infty} \frac{2x}{2x} = 1.$$

Thus $\lim_{n \to \infty} T_n = 1 \neq 0$ and so $\{F_n(x)\}$ does not converge uniformly on $[0, +\infty)$.

Question 2 (i). Since

$$\left| \frac{\cos nx}{n^2 + x^2} \right| \leq \frac{1}{n^2}$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the $p$-series, the series of functions $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + x^2}$ converges uniformly on $(-\infty, +\infty)$ by the Weierstrass $M$-test.

Question 2 (ii). Since

$$\left| \frac{1}{1 + n^3x^2} \right| \leq \frac{1}{1 + 2^2n^3} \leq \frac{1}{n^3}$$

for $x \geq 2$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by the $p$-series, the series of functions $\sum_{n=1}^{\infty} \frac{1}{1 + n^3x^2}$ converges uniformly on $[2, +\infty)$ by the Weierstrass $M$-test.
**Question 2 (iii).** Let $f_n(x) = \frac{xe^{-x}}{n^2}$ for $x \in (0, +\infty)$. Then

$$f'_n(x) = \frac{1}{n^2} \left( e^{-nx} - xne^{-nx} \right) = \frac{(1 - nx)e^{-nx}}{n^2}.$$ 

Thus $f_n(x)$ is monotone increasing for $0 \leq x \leq \frac{1}{n}$ and monotone decreasing for $x \geq \frac{1}{n}$. It follows that

$$|f_n(x)| \leq f_n \left( \frac{1}{n} \right) = \frac{\frac{1}{n}e^{-\frac{1}{n}}}{n^2} = \frac{e^{-1}}{n^3}$$

for $x \in (0, +\infty)$. Since the series $\sum_{n=1}^{\infty} \frac{e^{-1}}{n^3} = e^{-1} \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by the $p$-series, the series of functions $\sum_{n=1}^{\infty} \frac{xe^{-nx}}{n^2}$ converges uniformly on $(0, +\infty)$ by the Weierstrass $M$-test.

**Question 3.** Since the series of functions $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on $I$, by the Cauchy Criterion, for any $\epsilon > 0$, there exists $N$ such that

$$\left| \sum_{k=n+1}^{m} g_k(x) \right| < \epsilon$$

for all $x \in I$ and $m > n > N$. Because $g_k(x) \geq |f_k(x)| \geq 0$, for all $x \in I$ and $n > m > N$, we have

$$\left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=n+1}^{m} g_k(x) < \epsilon.$$ 

Thus, by the Cauchy Criterion, the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

**Question 4.** Answer: NO. Suppose that the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^x}$ converges uniformly on $(0, +\infty)$. By the Cauchy Criterion, for given $\epsilon = \frac{1}{2}$ there exists $N$ such that

$$\left| \sum_{k=n+1}^{m} (-1)^{n+1} \frac{1}{n^x} \right| < \frac{1}{2}$$
for all $n, m > N$ and $x \in (0, +\infty)$. In particular, let $n = N + 1$ and $m = N + 2$,

$$\frac{1}{(N + 2)^x} = \left| \sum_{k=N+2}^{N+2} (-1)^{n+1} \frac{1}{n^x} \right| < \frac{1}{2}$$

for all $x \in (0, +\infty)$. This contradicts to that

$$\frac{1}{(N + 2)^x} \geq \frac{1}{2}$$

when $0 < x \leq \frac{\ln 2}{\ln(N + 2)}$. \qed