INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of TWO (2) sections: Section A and Section B. It contains a total of SEVEN (7) questions and comprises FIVE (5) printed pages.

2. Answer ALL questions in Section A. Section A carries a total of 60 marks.

3. Answer no more than TWO (2) questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.
SECTION A

Answer all the questions in this section. Section A carries a total of 60 marks.

Question 1 [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) \[ \left\{ \sqrt{n^2 + 2n} - n \right\} . \]
(b) \[ \left\{ (6^n + 8^n)^{\frac{1}{n}} \right\} . \]
(c) \[ \left\{ \left( \frac{3n}{3n - 2} \right)^{2n + \sqrt{n}} \right\} . \]
(d) \[ \left\{ \frac{n^{100} \cdot 100^n \cdot \cos n}{n!} \right\} . \]

Question 2 [16 marks]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) \[ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 3n - 1} . \]
(b) \[ \sum_{n=1}^{\infty} \frac{1}{n(1 + 2 \ln n)} . \]
(c) \[ \sum_{n=1}^{\infty} 5^n \left( 1 - \frac{2}{n + 3} \right)^{n^2} . \]
(d) \[ \sum_{n=1}^{\infty} \frac{\sqrt{n + 1} - \sqrt{n}}{n} . \]
**Question 3** [10 marks]

Find the radius of convergence of each of the following power series. Justify your answer.

(a) \[ \sum_{k=1}^{\infty} \left( 1 - \frac{2}{k} \right)^{k^2} (x - 1)^k. \]

(b) \[ \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!} (2x + 1)^k. \]

**Question 4** [18 marks]

(a) Find limit inferior and limit superior of each of the following sequences.

(i) \[ \left\{ (-1)^n - \frac{1}{2} \right\}^n. \]

(ii) \[ \left\{ \left( 1 + \frac{1}{n} \right) \left( 1 + \sin \frac{n\pi}{4} \right)^\frac{1}{n} \right\}. \]

(b) Is the series \[ \sum_{n=1}^{\infty} (-1)^n \frac{2 \ln n + 1}{3\sqrt{n}} \] absolutely convergent, conditionally convergent or divergent? Justify your answer.
SECTION B
Answer not more than TWO (2) questions from this section. Each question in this section carries 20 marks.

**Question 5** [20 marks]

(a) Evaluate \( \lim_{n \to \infty} \int_{0}^{\frac{1}{2}} \frac{x^n \sin nx}{1 + x^n} \, dx \). Justify your answer.

(b) Find the interval of convergence of the power series \( \sum_{n=1}^{\infty} \frac{(1 - 2x)^n}{3n + 1} \).

(c) Let \( \{a_n\} \) be a bounded sequence of real numbers. Show that \( \lim_{n \to \infty} \sqrt[n]{|a_n|} \leq 1 \).

**Question 6** [20 marks]

(a) Consider the function \( f(x) = \sum_{n=1}^{\infty} x^n e^{-nx} \).

Is \( f(x) \) continuous on \([0, +\infty)\)? Justify your answer.

(b) Consider the sequence \( \{x_n\} \) defined recursively by

\[ x_1 = 2, \quad x_{n+1} = \frac{1}{2} \left( x_n + \frac{3}{x_n} \right), \text{ for } n \geq 1. \]

Show that \( \{x_n\} \) converges, and find its limit.

(c) Show that the series \( \sum_{k=1}^{\infty} \frac{1}{k^{1+2x}} \) does not converge uniformly on \((0, +\infty)\).
Question 7 [20 marks]

(a) Let $f(x) = x^5 \sin(x^9)$. Find $f^{(48)}(0)$.

(b) Let $A$ and $B$ be two non-empty bounded set of real numbers. Define $A + B = \{a + b \mid a \in A, \ b \in B\}$. Prove that $\inf A + \inf B = \inf(A + B)$.

(c) Show that the series of functions $\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n + 1)^x}$ converges uniformly on $[1, +\infty)$.