Question 1. [40 marks] Find limit inferior and limit superior of each of the following sequences.

(a) \( \left\{ (1 + (-1)^n) \sin \frac{n\pi}{4} \right\} \).

(b) \( \left\{ \frac{n + (-1)^n n^2}{n^2 + 1} \right\} \).

(c) \( \left\{ \lceil 1.5 + (-1)^n \rceil \right\} \).

(d) \( \left\{ \left( 1 + \frac{1}{n} \right) \left( 1 + \sin \frac{n\pi}{8} \right) \right\} \).

Question 2. [40 marks] Let \( \alpha > 0 \). Choose \( x_1 \geq \sqrt{\alpha} \). For \( n = 1, 2, 3, \ldots \), define \( x_{n+1} = \frac{1}{2} \left( x_n + \frac{\alpha}{x_n} \right) \).

(a) Show that the sequence \( \{x_n\} \) is bounded below by \( \sqrt{\alpha} \) and monotone decreasing.

(b) Prove that \( \lim_{n \to \infty} x_n = \sqrt{\alpha} \).

(c) Prove that \( 0 \leq x_n - \sqrt{\alpha} \leq \frac{x_n^2 - \alpha}{x_n} \).

(d) Let \( \alpha = 3 \) and \( x_1 = 2 \). Use part (c) to find \( x_n \) such that \( |x_n - \sqrt{3}| < 10^{-8} \).

Hint: From the inequality \( a^2 + b^2 \geq 2ab \), for \( y > 0 \), \( y + \frac{\alpha}{y} \geq 2\sqrt{\alpha} \cdot \sqrt{\frac{\alpha}{y}} = 2\sqrt{\alpha} \).

Question 3. [20 marks] Let \( \{a_n\} \) and \( \{b_n\} \) be bounded sequences in \( \mathbb{R} \). Prove that

\[ \liminf a_n + \limsup b_n \leq \limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n. \]