Question 1. [2 points, 1 for each part]
Prove the following limits by using \( \epsilon - N \) definition

i) \( \lim_{n \to \infty} \frac{3n + 8}{2n + 9} = \frac{3}{2} \).

ii) \( \lim_{n \to \infty} \frac{(-1)^n}{n^2 + 1} = 0 \).

Proof. (i). Note that

\[
\left| \frac{3n + 8 - 3}{2n + 9 - 2} \right| = \left| \frac{2(3n + 8) - 3(2n + 9)}{2(2n + 9)} \right| = \frac{11}{2(n + 9)} < \frac{11}{4n} < \frac{3}{n}.
\]

Given \( \epsilon > 0 \), choose \( N \) such that \( \frac{3}{N} \leq \epsilon \iff N \geq \frac{3}{\epsilon} \). When \( n > N \),

\[
\left| \frac{3n + 8 - 3}{2n + 9 - 2} \right| < \frac{3}{n} \leq \frac{3}{N} = \epsilon.
\]

(ii). Note that

\[
\left| \frac{(-1)^n}{n^2 + 1} - 0 \right| = \frac{n}{n^2 + 1} < \frac{n}{n^2} = \frac{1}{n}.
\]

Given \( \epsilon > 0 \), choose \( N \) such that \( \frac{1}{N} \leq \epsilon \iff N \geq \frac{1}{\epsilon} \). When \( n > N \),

\[
\left| \frac{(-1)^n}{n^2 + 1} - 0 \right| < \frac{1}{n} \leq \frac{1}{N} = \epsilon.
\]

\[ \square \]

Question 2. [5 points, 1 for each part]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) \( \left\{ \sqrt[n^2 + n - n]{2^n} \right\} \).

(b) \( \left\{ 2^n + 3^n \right\}^\frac{1}{n} \).

(c) \( \left\{ \sqrt[n! + 2n^5 + \ln n]{n! + 5n + 3n} \right\} \).

(d) \( \left\{ \frac{3n}{3n - 1} \right\}^{2n + \sqrt{n}} \).

(e) \( \left\{ \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} \right\} \).
Solution. (a).
\[
\lim_{n \to \infty} \left( \sqrt{n^2 + n - n} \right) = \lim_{n \to \infty} \frac{\sqrt{n^2 + n - n} \cdot \left( \sqrt{n^2 + n + n} \right)}{\sqrt{n^2 + n + n}}
\]
\[
= \lim_{n \to \infty} \frac{n^2 + n - n^2}{\sqrt{n^2 + n + n}} = \lim_{n \to \infty} \frac{n}{\sqrt{n^2 + n + n}} = \lim_{n \to \infty} \frac{1}{\sqrt{1 + 1/n + 1}} = \frac{1}{2}.
\]

(b).
\[
\lim_{n \to \infty} \left( 2^n + 3^n \right)^{\frac{1}{n}} = \lim_{n \to \infty} 3 \left[ \left( \frac{2}{3} \right)^n + 1 \right]^{\frac{1}{n}} = 3 \cdot (0 + 1)^0 = 3.
\]

Another solution: Since
\[
3 = \left( 3^n \right)^{\frac{1}{n}} \leq \left( 2^n + 3^n \right)^{\frac{1}{n}} \leq \left( 3^n + 3^n \right)^{\frac{1}{n}} = 2^{\frac{1}{n}} \cdot 3 \quad \text{and} \quad \lim_{n \to \infty} 2^{\frac{1}{n}} \cdot 3 = 3,
\]
\[
\lim_{n \to \infty} \left( 2^n + 3^n \right)^{\frac{1}{n}} = 3 \quad \text{by the Squeeze Theorem.}
\]

(c).
\[
\lim_{n \to \infty} \sqrt[n]{n! + 2n^5 + \ln n} = \lim_{n \to \infty} \sqrt[n]{\frac{1 + 2n^5 + \ln n}{1 + 5n + 3n}} = \sqrt[n]{1 + 0 + 0} = 1.
\]

(d).
\[
\lim_{n \to \infty} \left( \frac{3n}{3n - 1} \right)^{2n+\sqrt{n}} = \lim_{n \to \infty} \left( \frac{3n}{3n} \right)^{2n+\sqrt{n}} = \lim_{n \to \infty} \frac{1}{\left( 1 - \frac{1}{3n} \right)^{2n+\sqrt{n}}} = \frac{1}{\left( e^{-1/3} \right)^{\frac{3n}{\sqrt{n}}}} = e^{\frac{3}{2}}.
\]

(e). Note that
\[
\frac{n^{50} \cdot 50^n}{n!} \leq n^{50} \cdot 50^n \cdot \sin n \leq n^{50} \cdot 50^n.
\]
Since
\[
\lim_{n \to \infty} \frac{n^{50} \cdot 50^n}{n!} = \lim_{n \to \infty} \frac{n^{50} \cdot 50^n}{n!} = \lim_{n \to \infty} \frac{n^{50} \cdot 50^n}{n!} = 0 \cdot 0 = 0,
\]
\[
\lim_{n \to \infty} \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} = 0 \quad \text{by the Squeeze Theorem.}
\]

Question 3. [3 points, 1 for each part]

(a) If \( \{a_n\} \) is convergent, show that \( \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} a_n \).

(b) A sequence \( \{a_n\} \) is defined by \( a_1 = 1 \) and \( a_{n+1} = 1/(1 + a_n) \) for \( n \geq 1 \). Assume that \( \{a_n\} \) is convergent, find its limit.
(c) Find the limit of the sequence
\[ \{ \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2}\sqrt{2}}}, \ldots \} \).

Solution. (a). Let \( A = \lim_{n \to \infty} a_n \). Given any \( \epsilon > 0 \), there exists \( N \) such that
\[ |a_n - A| < \epsilon \]
for all \( n > N \) (by \( \epsilon - N \) definition). Write \( b_n \) for \( a_{n+1} \), that is \( b_n = a_{n+1} \). For all \( n > N \),
\[ |b_n - A| = |a_{n+1} - A| < \epsilon \]
because \( n + 1 > n > N \). From \( \epsilon - N \) definition, \( \lim \limits_{n \to \infty} b_n = A = \lim \limits_{n \to \infty} a_n \), that is,
\( \lim \limits_{n \to \infty} a_{n+1} = A = \lim \limits_{n \to \infty} a_n \).

(b). Let \( A = \lim \limits_{n \to \infty} a_n \). Then
\[ A = \lim \limits_{n \to \infty} a_{n+1} = \lim \limits_{n \to \infty} \frac{1}{1 + a_n} = \frac{1}{1 + \lim \limits_{n \to \infty} a_n} = \frac{1}{1 + A}. \]
Thus \( A(1 + A) = 1 \) or \( A^2 + A - 1 = 0 \). It follows that
\[ A = \frac{-1 \pm \sqrt{5}}{2}. \]
Next we show that \( a_n > 0 \) by induction. When \( n = 1, a_1 = 1 > 0 \). Suppose that \( a_n > 0 \). Then \( a_{n+1} = 1/(1 + a_n) > 0 \). The induction is finished and so \( a_n > 0 \) for all \( n \). It follows that \( A = \lim \limits_{n \to \infty} a_n \geq 0 \). The value \( \frac{-1 - \sqrt{5}}{2} \) is rejected because \( A \geq 0 \), and so
\[ A = \frac{-1 + \sqrt{5}}{2}. \]

(c). From the sequence, we see that \( a_1 = \sqrt{2} \) and \( a_{n+1} = \sqrt{2a_n} \). We show that \( \lim \limits_{n \to \infty} a_n \) exists:
First we prove by induction that \( 0 \leq a_n \leq 2 \). When \( n = 1, 0 \leq a_1 \leq 2 \) holds. Suppose that \( 0 \leq a_n \leq 2 \). Then
\[ 0 \leq \sqrt{2a_n} = a_{n+1} \leq \sqrt{2 \cdot 2} = 2. \]
The induction is finished and so \( 0 \leq a_n \leq 2 \) for all \( n \).
Next since \( 0 \leq a_n \leq 2 \), \( a_{n+1} = \sqrt{2a_n} \geq \sqrt{a_n \cdot a_n} = a_n \) for all \( n \). Thus \( \{a_n\} \) is monotone increasing. By monotone convergence theorem, \( \lim \limits_{n \to \infty} a_n \) exists.
Finally let \( A = \lim \limits_{n \to \infty} a_n \). Then
\[ A = \lim \limits_{n \to \infty} a_{n+1} = \sqrt{2 \lim \limits_{n \to \infty} a_n} = \sqrt{2A} \]
and so \( A^2 = 2A \) or \( A = 0, 2 \). Since \( \{a_n\} \) is monotone increasing, \( a_n \geq a_1 = \sqrt{2} \) for all \( n \). It follows that \( A = \lim \limits_{n \to \infty} a_n \geq \sqrt{2} \). Thus \( 0 \) is rejected and so \( A = 2 \). □