Question 1 (a). Let $a_n = \frac{n^2 - 1}{2n^2 + n}$. Then $\lim_{n \to \infty} a_n = \frac{1}{2} \neq 0$ and so the series $\sum_{n=1}^{\infty} \frac{n^2 - 1}{2n^2 + n}$ is divergent by the divergence test.

Question 1 (b). Let $a_n = \sin \frac{n\pi}{2}$. Then $\{a_n\} = \{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$ and so $\lim_{n \to \infty} a_n$ does not exist. Thus the series $\sum_{n=1}^{\infty} \sin \frac{n\pi}{2}$ is divergent by the divergence test.

Question 1 (c). Let $a_n = \frac{n^2 + 1 + \ln n}{n + n^3 + 4}$ and let $b_n = \frac{1}{n}$. Then

$$\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{n + n^3 + 4}{n^2 + 1 + \ln n} = \lim_{n \to \infty} \frac{1}{n^2} \cdot \frac{1 + \frac{4}{n^2} + \frac{1}{n^2}}{1 + \frac{1}{n^2} + \frac{\ln n}{n^2}} = \frac{0 + 1 + 0}{1 + 0 + 0} = 1.$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, so is $\sum_{n=1}^{\infty} \frac{n^2 + 1 + \ln n}{n + n^3 + 4}$ by the limit comparison test.

Question 1 (d). Observe that $3 + \sin \frac{n}{n^2} \leq \frac{4}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{4}{n^2}$ is convergent by the $p$-series, the positive series $\sum_{n=1}^{\infty} \frac{3 + \sin \frac{n}{n^2}}{n^2}$ is convergent by the comparison test.

Question 1 (e). Observe that

$$\frac{2^n + 3}{3^{n+1} - n} \leq \frac{2^n + 2^n}{3^{n+1} - n} = \frac{2^{n+1}}{3^n} = 2 \left(\frac{2}{3}\right)^n$$

for $n \geq 2$. Since $\sum_{n=1}^{\infty} 2 \left(\frac{2}{3}\right)^n$ is convergent by the geometric series, the positive series $\sum_{n=1}^{\infty} \frac{2^n + 3}{3^{n+1} - n}$ is convergent by the comparison test.

Question 1 (f). Let $a_n = \frac{2}{n^{1+\frac{1}{3}}}$ and let $b_n = \frac{1}{n}$. Observe that

$$\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{n^{1+\frac{1}{3}}}{2} = \lim_{n \to \infty} \frac{\sqrt[3]{n}}{2} = \frac{1}{2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series $\sum_{n=1}^{\infty} \frac{2}{n^{1+\frac{1}{3}}}$ is divergent by the limit comparison test.
Question 1 (g). Observe that

\[ \frac{4 + (-1)^n}{2n} \geq \frac{3}{2n}. \]

Since \( \sum_{n=1}^{\infty} \frac{3}{2n} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{4 + (-1)^n}{2n} \) is divergent by the comparison test. \( \square \)

Question 1 (h). Observe that

\[ \frac{1}{n(1+\ln n)^p} = \frac{(1+\ln n)^{-p}}{n} \geq \frac{1}{n} \]

for \( p \leq 0 \). Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)^p} \) is divergent for \( p \leq 0 \) by the comparison test. \( \square \)

Question 1 (i). Observe that

\[ \frac{n}{n^2 + 1} \geq \frac{n}{2n^2 + n^2} = \frac{n}{2n^2} = \frac{1}{2n}. \]

Since \( \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \) is divergent by the comparison test. \( \square \)

Question 2(a). Since \( \sum_{n=1}^{\infty} a_n \) is convergent, we have \( \lim_{n \to \infty} a_n = 0 \) and so there exists a positive integer \( N \) such that \( a_n = |a_n| = |a_n - 0| < 1 \) for \( n > N \). Since \( a_n \geq 0 \),

\[ a_n^2 = a_n \cdot a_n \leq 1 \cdot a_n = a_n \]

for \( n > N \). By the comparison test, the positive series \( \sum_{n=1}^{\infty} a_n^2 \) is convergent. \( \square \)

Another Solution of Question 2 (a). Since \( \sum_{n=1}^{\infty} a_n \) is convergent, we have \( \lim_{n \to \infty} a_n = 0 \). Let \( b_n = a_n^2 \). Then

\[ \lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} a_n = 0, \]

that is, \( b_n \ll a_n \). Since \( a_n \geq 0 \), by limit comparison test, \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n^2 \) is convergent. \( \square \)
Question 2 (b). Let \( a_n = \frac{1}{n^2} \). Then \( \sum_{n=1}^{\infty} a_n \) is convergent but \( \sum_{n=1}^{\infty} \sqrt{a_n} \) is divergent by the \( p \)-series.

Question 3 (a). Let \( f(x) = \frac{1}{x(1 + \ln x)} \). Then \( f(x) \) is a positive monotone decreasing function over \([1, +\infty)\). Since

\[
\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x(1 + \ln x)}\,dx = \int_0^{\infty} \frac{1}{1 + y}\,dy = \ln(1 + y)|_0^\infty = +\infty.
\]

is divergent, the series \( \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)} \) is divergent by the integral test.

Question 3 (b). Let \( f(x) = \frac{1}{x[1 + (\ln x)^2]} \). Then \( f(x) \) is a positive monotone decreasing function over \([1, +\infty)\). Since

\[
\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x[1 + (\ln x)^2]}\,dx = \int_0^{\infty} \frac{1}{1 + y^2}\,dy = \arctan y|_0^\infty = \frac{\pi}{2}
\]

is convergent, the series \( \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)} \) is convergent by the integral test.