Question 1 (a). The limiting function $F(x) = \lim_{n \to \infty} \frac{n^2}{n^2 + x^2} = \lim_{n \to \infty} \frac{1}{1 + \frac{x^2}{n^2}} = 1.$

\[0 \leq T_n = \sup_{0 \leq x \leq 1} |F_n(x) - F(x)| = \sup_{0 \leq x \leq 1} \left| \frac{n^2}{n^2 + x^2} - 1 \right| = \sup_{0 \leq x \leq 1} \frac{x^2}{n^2 + x^2} \leq \frac{1}{n^2}.
\]

Since $\lim_{n \to \infty} \frac{1}{n^2} = \lim_{n \to \infty} 0 = 0,$ we have $\lim_{n \to \infty} T_n = 0$ by the Squeeze theorem and so $\{F_n\}$ converges uniformly on $[0, 1].$ \hfill \Box

Question 1 (b). The limiting function $F(x) = \lim_{n \to \infty} x^n(1 - x) = 0$ for $0 \leq x \leq 1.$ Observe

\[T_n = \sup_{0 \leq x \leq 1} |F_n(x) - F(x)| = \sup_{0 \leq x \leq 1} x^n(1 - x).
\]

Let $g(x) = x^n(1 - x).$ Then $g'(x) = nx^{n-1}(1 - x) - x^n = x^{n-1}[n - (n + 1)x].$ From $g'(x) = 0,$ we have $x = 0$ or $\frac{n}{n+1}.$ Since $g'(x) \geq 0$ for $0 \leq x \leq \frac{n}{n+1}$ and $g'(x) \leq 0$ for $\frac{n}{n+1} \leq x \leq 1,$ $\sup_{0 \leq x \leq 1} g(x) = \max\{g(x) | 0 \leq x \leq 1\} = \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right)$ and so

\[\lim_{n \to \infty} T_n = \lim_{n \to \infty} \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) = \lim_{n \to \infty} \frac{1}{(1 + \frac{1}{n})^n} \cdot \frac{1}{n+1} = \frac{1}{e} \cdot 0 = 0.
\]

Thus $\{F_n\}$ converges uniformly on $[0, 1].$ \hfill \Box

Question 1 (c). The limiting function $f(x) = \lim_{n \to \infty} \frac{n \ln x}{x^n} = 0$ for $1 \leq x < \infty.$ Observe that

\[T_n = \sup_{x \geq 1} |f_n(x) - f(x)| = \sup_{x \geq 1} \left| \frac{n \ln x}{x^n} - 0 \right| = \sup_{x \geq 1} \frac{n \ln x}{x^n}.
\]

Let $g(x) = \frac{n \ln x}{x^n}.$ From

\[g'(x) = \left(n \ln x \cdot x^{-n}\right)' = \frac{1}{x} x^{-n} - n^2 \ln x \cdot x^{-n} = \frac{n - n^2 \ln x}{x^{n+1}} = 0,
\]

we have $x = e^{\frac{1}{n}}.$ Since $g'(x) \geq 0$ for $1 \leq x \leq e^{\frac{1}{n}}$ and $g'(x) \leq 0$ for $x \geq e^{\frac{1}{n}},$ we have

\[T_n = \sup_{x \geq 1} g(x) = \max\{g(x) | x \geq 1\} = \frac{n \ln e^{\frac{1}{n}}}{\left(e^{\frac{1}{n}}\right)^n} = \frac{1}{e}
\]

and so $\lim_{n \to \infty} T_n = \frac{1}{e} \neq 0.$ Thus $\{f_n\}$ does not converge uniformly on $[1, +\infty).$ \hfill \Box
Question 1 (d). The limiting function \( f(x) = \lim_{n \to \infty} \frac{n \ln x \cos nx}{x^n} = 0 \) for \( x \geq 4 \). Observe that

\[
0 \leq T_n = \sup_{x \geq 4} |f_n(x) - f(x)| = \sup_{x \geq 4} \frac{n \ln x \cdot |\cos nx|}{x^n} \leq \sup_{x \geq 4} \frac{n \ln x}{x^n} = g(4) = \frac{n \ln 4}{4^n},
\]

where \( g(x) = \frac{n \ln x}{x^n} \) is monotone decreasing on \([4, +\infty)\). Since \( \lim_{n \to \infty} \frac{n \ln 4}{4^n} = \lim_{n \to \infty} 0 = 0 \), we have \( \lim_{n \to \infty} T_n = 0 \) by the Squeeze theorem and so \( \{f_n\} \) converges uniformly on \([4, +\infty)\). □

Question 1 (e). The limiting function \( F(x) = \lim_{n \to \infty} F_n(x) = \lim_{n \to \infty} \frac{n^2}{n^2 + x^2} = 1 \). Observe that

\[
T_n = \sup_{x \geq 0} |F_n(x) - F(x)| = \sup_{x \geq 0} \left| \frac{n^2}{n^2 + x^2} - 1 \right| = \sup_{x \geq 0} \frac{x^2}{n^2 + x^2}.
\]

Let \( g(x) = \frac{x^2}{n^2 + x^2} \). Since

\[
g'(x) = \frac{2x(n^2 + x^2) - x^2 \cdot 2x}{(n^2 + x^2)^2} = \frac{2xn^2}{(n^2 + x^2)^2} \geq 0
\]

for \( x \geq 0 \), the function \( g(x) \) is monotone increasing and so

\[
T_n = \sup_{x \geq 0} g(x) = \lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{x^2}{n^2 + x^2} = \lim_{x \to \infty} \frac{2x}{2x} = 1.
\]

Thus \( \lim_{n \to \infty} T_n = 1 \neq 0 \) and so \( \{F_n(x)\} \) does not converge uniformly on \([0, +\infty)\). □

Question 2 (i). Since

\[
\left| \frac{\cos nx}{n^2 + x^2} \right| \leq \frac{1}{n^2}
\]

and the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent by the \( p \)-series, the series of functions \( \sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + x^2} \) converges uniformly on \((-\infty, +\infty)\) by the Weierstrass \( M \)-test. □

Question 2 (ii). Since

\[
\left| \frac{1}{1 + n^3x^2} \right| \leq \frac{1}{1 + 2^3n^3} \leq \frac{1}{n^3}
\]

for \( x \geq 2 \) and the series \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) is convergent by the \( p \)-series, the series of functions \( \sum_{n=1}^{\infty} \frac{1}{1 + n^3x^2} \) converges uniformly on \([2, +\infty)\) by the Weierstrass \( M \)-test. □
Question 2 (iii). Let \( f_n(x) = \frac{xe^{-x}}{n^2} \) for \( x \in (0, +\infty) \). Then
\[
f'_n(x) = \frac{1}{n^2} \left( e^{-nx} - xne^{-nx} \right) = \frac{(1 - nx)e^{-nx}}{n^2}.
\]
Thus \( f_n(x) \) is monotone increasing for \( 0 \leq x \leq \frac{1}{n} \) and monotone decreasing for \( x \geq \frac{1}{n} \). It follows that
\[
|f_n(x)| \leq f_n \left( \frac{1}{n} \right) = \frac{\frac{1}{n}e^{-\frac{1}{n^2}}}{n^2} = \frac{e^{-1}}{n^3}
\]
for \( x \in (0, +\infty) \). Since the series \( \sum_{n=1}^{\infty} \frac{e^{-1}}{n^3} = e^{-1} \sum_{n=1}^{\infty} \frac{1}{n^3} \) is convergent by the \( p \)-series, the series of functions \( \sum_{n=1}^{\infty} \frac{xe^{-nx}}{n^2} \) converges uniformly on \( (0, +\infty) \) by the Weierstrass \( M \)-test. \( \square \)

Question 3. Since the series of functions \( \sum_{n=1}^{\infty} g_n(x) \) converges uniformly on \( I \), by the Cauchy Criterion, for any \( \epsilon > 0 \), there exists \( N \) such that
\[
\sum_{k=n+1}^{m} g_k(x) = \left| \sum_{k=n+1}^{m} g_k(x) \right| < \epsilon
\]
for all \( x \in I \) and \( m > n > N \). Because \( g_k(x) \geq |f_k(x)| \geq 0 \), for all \( x \in I \) and \( n > m > N \), we have
\[
\left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=n+1}^{m} g_k(x) < \epsilon.
\]
Thus, by the Cauchy Criterion, the series of functions \( \sum_{n=1}^{\infty} f_n(x) \) converges uniformly. \( \square \)

Question 4. Answer: NO. Suppose that the series \( \sum_{n=1}^{\infty} (\frac{-1}{n^2} \frac{1}{n^2}) \) converges uniformly on \( (0, +\infty) \). By the Cauchy Criterion, for given \( \epsilon = \frac{1}{2} \) there exists \( N \) such that
\[
\left| \sum_{k=n+1}^{m} (-1)^{n+1} \frac{1}{n^2} \right| < \frac{1}{2}
\]
for all $n, m > N$ and $x \in (0, +\infty)$. In particular, let $n = N + 1$ and $m = N + 2$,

$$\frac{1}{(N + 2)^x} = \left| \sum_{k=N+2}^{N+2} (-1)^{n+1} \frac{1}{n^x} \right| < \frac{1}{2}$$

for all $x \in (0, +\infty)$. This contradicts to that

$$\frac{1}{(N + 2)^x} \geq \frac{1}{2}$$

when $0 < x \leq \frac{\ln 2}{\ln(N + 2)}$. \hfill \Box