In order to get the full marks for the project of take-home exams, you need to have 50 or more points. Well there are 10 points in this take-home exam.

**Question 1** [2 points, 1 for each part]

Let $a_1$ and $b_1$ be positive numbers with $a_1 > b_1$. Let $a_2 = \frac{a_1 + b_1}{2}$ be their arithmetic mean and let $b_2 = \sqrt{a_1 b_1}$ be their geometric mean. Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}.$$

(a) Show by mathematical induction that $a_n > a_{n+1} > b_{n+1} > b_n$.

(b) Prove that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$.

(Note. Gauss called this the common value of these limits the **arithmetic-geometric mean** of the numbers $a = a_1$ and $b = b_1$.)

**Question 2.** [3 points, 1 for each part]

Find limit inferior and limit superior of each of the following sequences.

(a) $\left\{ \frac{n + (-1)^n n^2}{n^2 + 1} \right\}$.

(b) $\left\{ 1.5 + (-1)^n \right\}$. 

(c) $\left\{ \left( 1 + \frac{1}{n} \right) \left( 1 + \sin \frac{n \pi}{8} \right)^{\frac{1}{n}} \right\}$

**Question 3** [5 points, 1 for each part]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2 + 2k - 1}$.

(b) $\sum_{n=1}^{\infty} \frac{1}{n(2 + \ln n)}$.

(c) $\sum_{n=1}^{\infty} 6^n \left( 1 - \frac{2}{n+1} \right)^{n^2}$.

(d) $\sum_{n=1}^{\infty} \frac{n^n}{3^n \cdot n!}$.

(e) $\sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{k}$.