Take-home Exam 5

In order to get the full marks for the project of take-home exams, you need to have 50 or more points. Well there are 10 points in this take-home exam.

**Question 1.** [6 points, 1 for each part] Determine whether the following series of functions converge uniformly on the indicated intervals. Justify your answers.

(a) \( \sum_{k=1}^{\infty} \frac{k \sin kx}{k^3 + x^2}, \ x \in [0, \infty). \)

(b) \( \sum_{k=1}^{\infty} e^{-kx}x^k, \ x \in [0, \infty). \)

(c) \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k + x}, \ x \in [0, \infty). \)

Hint: Let \( S(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k + x} \) and let \( S_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k + x} \). By using alternating series estimation, show that \( T_n = \sup_{x \geq 0} |S_n(x) - S(x)| \leq \sup_{x \geq 0} \frac{1}{n + 1 + x} \leq \frac{1}{n + 1}. \)

(d) \( \sum_{k=1}^{\infty} \frac{x^k}{1 + k(\ln k)^2}, \ x \in [-1, 1]. \)

(e) \( \sum_{n=0}^{\infty} \left( \frac{1}{nx + 2} - \frac{1}{(n+1)x + 2} \right), \ x \in [0, 1]. \)

(Hint: Find the partial sums \( S_n(x) = \sum_{k=0}^{n} \left( \frac{1}{kx + 2} - \frac{1}{(k+1)x + 2} \right) \) and then show that \( S(x) = \lim_{n \to \infty} S_n(x) \) is not continuous while each \( S_n(x) \). From this, conclude that the series of functions does not converge uniformly.)

(f) \( \sum_{k=1}^{\infty} \left( \frac{x^k}{2} \right), \ x \in (-2, 2). \)

(Hint: Try T-test. Show that \( T_n = \sup_{-2 < x < 2} \left| \sum_{k=n+1}^{\infty} \left( \frac{x}{2} \right)^k \right| = +\infty \) by letting \( x \to 2^{-}. \))

**Question 2** [4 points, 1 for each part] Find the radius of convergence of each of the following power series:

(a) \( \sum_{k=1}^{\infty} \frac{3^k}{k^3}(2x + 1)^k. \)

(b) \( \sum_{k=1}^{\infty} \left( 1 - \frac{1}{k} \right)^{k^2} (x - 1)^k. \)
\( (c) \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!} x^k \)

\( (d) \sum_{n=0}^{\infty} n \left( \frac{x}{2} \right)^{n^2} \).

(Hint. For part (d), you can use the root test to find the interval for which the series converges absolutely.)