**Question 1 (a).** Let \( a_n = \frac{n^2 - 1}{2n^2 + n} \). Then \( \lim_{n \to \infty} a_n = 1 / 2 \neq 0 \) and so the series \( \sum_{n=1}^{\infty} \frac{n^2 - 1}{2n^2 + n} \) is divergent by the divergence test.

**Question 1 (b).** Let \( a_n = \sin \frac{n\pi}{2} \). Then \( \{a_n\} = \{1, 0, -1, 0, 1, 0, -1, 0, \ldots\} \) and so \( \lim_{n \to \infty} a_n \) does not exist. Thus the series \( \sum_{n=1}^{\infty} \sin \frac{n\pi}{2} \) is divergent by the divergence test.

**Question 1 (c).** Let \( a_n = \frac{n^2 + 1 + \ln n}{n + n^3 + 4} \) and let \( b_n = \frac{1}{n} \). Then

\[
\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{n + n^3 + 4}{n^2 + 1 + \ln n} = \lim_{n \to \infty} \frac{1 + \frac{1}{n^2} + \frac{4}{n^3}}{1 + \frac{1}{n^2} + \ln \frac{1}{n}} = 0 + 0 + 0 = 1.
\]

Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent, so is \( \sum_{n=1}^{\infty} \frac{n^2 + 1 + \ln n}{n + n^3 + 4} \) by the limit comparison test.

**Question 1 (d).** Observe that \( 3 + \sin \frac{n\pi}{2} \leq 4 \). Since \( \sum_{n=1}^{\infty} \frac{4}{n^2} \) is convergent by the \( p \)-series, the positive series \( \sum_{n=1}^{\infty} \frac{3 + \sin \frac{n\pi}{2}}{n^2} \) is convergent by the comparison test.

**Question 1 (e).** Observe that

\[
\sum_{n=1}^{\infty} \frac{2^n + 3}{3^{n+1} - n} \leq 2 \left( \frac{2}{3} \right)^n
\]

for \( n \geq 2 \). Since \( \sum_{n=1}^{\infty} 2 \left( \frac{2}{3} \right)^n \) is convergent by the geometric series, the positive series \( \sum_{n=1}^{\infty} \frac{2^n + 3}{3^{n+1} - n} \) is convergent by the comparison test.

**Question 1 (f).** Let \( a_n = \frac{2}{n^{1+\pi}} \) and let \( b_n = \frac{1}{n} \). Observe that

\[
\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{n^{1+\frac{1}{\pi}}}{2} = \lim_{n \to \infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}.
\]

Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{2}{n^{1+\pi}} \) is divergent by the limit comparison test.
Question 1 (g). Observe that
\[
\frac{4 + (-1)^n}{2n} \geq \frac{3}{2n}.
\]
Since
\[
\sum_{n=1}^{\infty} \frac{3}{2n} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n}
\]
is divergent by the harmonic series, the positive series
\[
\sum_{n=1}^{\infty} \frac{4 + (-1)^n}{2n}
\]
is divergent by the comparison test. □

Question 1 (h). Observe that
\[
\frac{1}{n(1 + \ln n)^p} = \frac{(1 + \ln n)^{-p}}{n} \geq \frac{1}{n}
\]
for \( p \leq 0 \). Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series
\[
\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)^p}
\]
is divergent for \( p \leq 0 \) by the comparison test. □

Question 1 (i). Observe that
\[
\frac{n}{n^2 + 1} \geq \frac{n}{2n^2 + n^2} = \frac{n}{2n^2} = \frac{1}{2n}.
\]
Since \( \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series
\[
\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}
\]
is divergent by the comparison test. □

Question 2(a). Since \( \sum_{n=1}^{\infty} a_n \) is convergent, we have \( \lim_{n \to \infty} a_n = 0 \) and so there exists a positive integer \( N \) such that \( a_n = |a_n| = |a_n - 0| < 1 \) for \( n > N \). Since \( a_n \geq 0 \),
\[
a_n^2 = a_n \cdot a_n \leq 1 \cdot a_n = a_n
\]
for \( n > N \). By the comparison test, the positive series \( \sum_{n=1}^{\infty} a_n^2 \) is convergent. □

Another Solution of Question 2 (a). Since \( \sum_{n=1}^{\infty} a_n \) is convergent, we have \( \lim_{n \to \infty} a_n = 0 \).
Let \( b_n = a_n^2 \). Then
\[
\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} a_n = 0,
\]
that is, \( b_n \ll a_n \). Since \( a_n \geq 0 \), by limit comparison test, \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n^2 \) is convergent. □
Question 2 (b). Let \( a_n = \frac{1}{n^2} \). Then \( \sum_{n=1}^{\infty} a_n \) is convergent but \( \sum_{n=1}^{\infty} \sqrt{a_n} \) is divergent by the p-series.

\[ \] □

Question 3 (a). Let \( f(x) = \frac{1}{x(1 + \ln x)} \). Then \( f(x) \) is a positive monotone decreasing function over \([1, +\infty)\). Since

\[
\int_{1}^{\infty} f(x) \, dx = \int_{1}^{\infty} \frac{1}{x(1 + \ln x)} \, dx = \left. \int_{0}^{\infty} \frac{1}{1 + y} \, dy \right|_{0}^{\infty} = +\infty.
\]

is divergent, the series \( \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)} \) is divergent by the integral test. □

Question 3 (b). Let \( f(x) = \frac{1}{x[1 + (\ln x)^2]} \). Then \( f(x) \) is a positive monotone decreasing function over \([1, +\infty)\). Since

\[
\int_{1}^{\infty} f(x) \, dx = \int_{1}^{\infty} \frac{1}{x[1 + (\ln x)^2]} \, dx = \left. \int_{0}^{\infty} \frac{1}{1 + y^2} \, dy \right|_{0}^{\infty} = \frac{\pi}{2}
\]

is convergent, the series \( \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)} \) is convergent by the integral test. □