From Braid Groups to Homotopy Groups

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• Joint with Jon Berrick (Singapore), Fred Cohen (Rochester), Yan Loi Wong (Singapore)


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Outline

Homotopy Groups

Braid Groups

Brunnian Braids

Theorem

Methods of Proof
Homotopy Group

- $\pi_n(X) := [S^n, X]$, the set of the (pointed) homotopy classes of (pointed) continuous maps from the n-sphere $S^n$ to $X$.

- $\pi_0(X)$ is the set of path-connected components of $X$, which is not a group in general.

- fundamental group $\pi_1(X)$ is a group, but non-commutative in general.

- $\pi_n(X)$ is an abelian group for $n \geq 2$. 
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Remarks

- Čech defined the higher homotopy groups, but abandoned them they are abelian. (1930s)

- It was originally conjectured that the homotopy groups of spheres are isomorphic to their homology groups. Then Heinz Hopf invented the Hopf map.

- Applications: Classification of vector bundles, fibre bundles, Algebraic $K$-theory, deformation theory, mathematical physics and etc.

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Examples

• \( \pi_n(S^1) = 0 \) for \( n \neq 1 \) and \( \pi_1(S^1) = \mathbb{Z} \).

• For \( n > 0 \), \( \pi_m(S^n) = 0 \) for \( m < n \) and \( \pi_n(S^n) = \mathbb{Z} \).

• Curtis proved that \( \pi_i(S^5) \neq 0 \) for all \( i \geq 5 \).

• \( \pi_m(S^n) \) for \( m > n \) is not yet well understood for general \( m \) and \( n \geq 2 \), although many non-zero elements are known.
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Main (traditional) methods in calculating $\pi_\ast(S^n)$:

- **EHP sequence**: long exact sequence for giving relations between the homotopy groups of different spheres;

- Toda’s brackets: Operations on $\pi_\ast(S^n)$;

- Adams spectral sequence: General homological method;

- Morava $K$-theory and periodic elements: families of special elements in $\pi_\ast(S^n)$.

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Our ideas:

- **Step 1.** Describe the homotopy groups as the derived groups of the braid groups, that is, the quotient of certain subgroup of the braid group by another subgroup.
  
  - this step seems pretty successful, namely there are good (canonical) descriptions of the homotopy groups as the derived groups of the braids that I am going to talk today.

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History of Braids

- traditional: every traditional woman knows well how to make braids. **Watch their hair.**

- 1890’s: Braids and Links, Hurwicz (1891), Brunn (1892), Fricke and Klein (1897)

- 1925: Artin

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configuration spaces

- **ordered configuration space**
  \[ F(M, n) = \{(x_1, x_2, \ldots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\} \]

- **unordered configuration space**
  \[ B(M, n) = F(M, n)/\Sigma_n. \]

- The covering map \( p: F(M, n) \longrightarrow B(M, n) \) with fibre \( \Sigma_n \).
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$n$-strand braid group over $M$

- $B_n(M) = \pi_1(B(M, n))$

intuitive description:

- Choose a base point $(q_1, q_2, \cdots, q_n)$ for $F(M, n)$.
- Let $\omega: S^1 \to B(M, n)$ be a loop.
- Then there is a lifting path $\lambda: [0, 1] \to F(M, n)$ such that $\lambda(0) = (q_1, q_2, \cdots, q_n), \lambda(1) = (q_{\sigma(1)}, \cdots, q_{\sigma(n)})$ for some $\sigma \in \Sigma_n$ and $p(\lambda) = \omega$.
- Thus $\lambda(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t))$ with $\lambda_i(t) \neq \lambda_j(t)$ for $i \neq j$ and $0 \leq t \leq 1$.
- We obtain $n$-strand $\lambda_i(t)$ in the cylinder $M \times I$ starting at $q_i$ and ending with $q_{\sigma(i)}$ for some $\sigma$.
- The multiplication is given by the composition of strands.
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$n$-strand Pure Braid Group over $M$

- $P_n(M) = \pi_1(F(M, n))$

- In other words, the pure braids are $n$ strands $\lambda_i(t)$ in $M \times I$ starting at $q_i$ and ending with $q_i$.

- When $M$ is the unit disk $D^2$, $B_n = B_n(D^2)$ is the classical Artin braid group.

- Any link can be obtained by closing up an (Artin) braid.
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Face operations on braids

- Consider the **coordinate projections** $d_i : F(M, n + 1) \rightarrow F(M, n)$
  $$(x_0, x_1, \ldots, x_n) \mapsto (x_0, x_1, \ldots, \hat{x}_i, \ldots, x_n).$$

- The map $d_i$ induces, by taking the fundamental group, a group homomorphism $d_i = d_{i*} : P_{n+1}(M) \rightarrow P_n(M)$ and a function $d_i : B_{n+1}(M) \rightarrow B_n(M)$ given by
  $$(\lambda_0(t), \ldots, \lambda_n(t)) \mapsto (\lambda_0(t), \ldots, \hat{\lambda}_i(t), \ldots, \lambda_n(t)),$$
  that is, **deleting the $(i + 1)$-st strand for $0 \leq i \leq n$.**
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Brunnian Braids

- A braid \( \beta \in B_{n+1}(M) \) is called \textbf{Brunnian} if \( d_i(\beta) = 1 \) for all \( 0 \leq i \leq n \).

- In other words, the group of Brunnian braids \( \text{Brun}_{n+1}(M) \) is given by

\[
\text{Brun}_{n+1}(M) : = \bigcap_{i=0}^{n} \text{Ker}(d_i : B_{n+1}(M) \to B_n(M)).
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- The classical \textbf{Borromean Rings} is a link by closing up a Brunnian braid of 3 strings over \( D^2 \).
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Theorem

• The canonical embedding $f: D^2 \subseteq S^2$ induces a group homomorphism $\text{Brun}_n(D^2) \xrightarrow{f_*} \text{Brun}_n(S^2)$.

• Theorem. There is an exact sequence of groups

$$\text{Brun}_{n+1}(S^2) \hookrightarrow \text{Brun}_n(D^2) \xrightarrow{f_*} \text{Brun}_n(S^2) \twoheadrightarrow \pi_{n-1}(S^2)$$

for $n \geq 5$.

• The image of $f_*: \text{Brun}_n(D^2) \rightarrow \text{Brun}_n(S^2)$ is a normal subgroup.

• Both $\text{Brun}_n(D^2)$ and $\text{Brun}_n(S^2)$ are free groups of infinite rank for $n \geq 5$. 
Theorem

- The canonical embedding $f : D^2 \subseteq S^2$ induces a group homomorphism $\text{Brun}_n(D^2) \xrightarrow{f_*} \text{Brun}_n(S^2)$.

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Examples

• For instance, $\text{Brun}_5(S^2)$ modulo $\text{Brun}_5(D^2)$ is $\pi_4(S^2) = \mathbb{Z}/2$.

• The other low homotopy groups of $S^2$ are as follows:
  
  \begin{align*}
  \pi_5(S^2) &= \mathbb{Z}/2, \\
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  \pi_7(S^2) &= \mathbb{Z}/2, \\
  \pi_8(S^2) &= \mathbb{Z}/2, \\
  \pi_9(S^2) &= \mathbb{Z}/3, \\
  \pi_{10}(S^2) &= \mathbb{Z}/15, \text{ and etc.}
  \end{align*}

• Thus, up to certain range, $\text{Brun}_{n+1}(S^2)$ modulo $\text{Brun}_{n+1}(D^2)$ are known by non-trivial calculations of $\pi_*(S^2)$. 
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Remarks


- If her old question were answered, then, together with some of my works, one has the combinational determination of the homotopy groups $\pi_n(S^2)$ by listing generators and relations.

- **Problem:** Determine the order of Brun$_n$(S$^2$)/Brun$_n$(D$^2$), which is finite for $n \geq 5$.
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• The functions $d_i : B_{n+1}(M) \to B_n(M)$, obtained by deleting $i + 1$st stand for $0 \leq i \leq n$, satisfy the following identity:
  
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Methods of Proof

- A **Δ-set** (Δ-group) means a sequence of sets (groups) $S = \{S_n\}_{n \geq 0}$ with face functions (face homomorphisms) $d_i : S_n \to S_{n-1}$ such that the above identity holds.

- A simplicial group means a Δ-group $G = \{G_n\}_{n \geq 0}$ together with degeneracy homomorphisms $s_i : G_n \to G_{n+1}$ such that so-called simplicial identities hold.

- The theorem is obtained by
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