INTRODUCTION TO ALGEBRAIC TOPOLOGY
TUTORIAL 1

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Exercise 0.1. a) Show that each of the following is a metric for \( \mathbb{R}^n \):

\[ d(x, y) = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} = ||x - y||; \quad d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y; \end{cases} \]

\[ d(x, y) = \sum_{i=1}^{n} |x_i - y_i|; \quad d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|. \]

b) Show that \( d(x, y) \) does not define a metric on \( \mathbb{R} \).

c) Show that \( d(x, y) = \min_{1 \leq i \leq n} |x_i - y_i| \) does not define a metric on \( \mathbb{R}^n \).

d) Let \( d \) be a metric. Show that \( d' \) defined by

\[ d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} \]

is also a metric.

Exercise 0.2. Show that if \( \mathcal{U} \) is the family of open sets arising from a metric space then

i) The empty set \( \emptyset \) and the whole set belong to \( \mathcal{U} \);

ii) The intersection of two members of \( \mathcal{U} \) belongs to \( \mathcal{U} \);

iii) The union of any number of members of \( \mathcal{U} \) belongs to \( \mathcal{U} \).

Exercise 0.3. Let \( \mathcal{U} \) be a topology for \( X \). Show that the intersection of a finite number of members of \( \mathcal{U} \) is in \( \mathcal{U} \). Show by examples that infinite intersection of open sets in a topological space may not be open.

Exercise 0.4. Let \( X \) be a metric space with metric \( d \). Let \( d' \) be the new metric defined in Exercise 0.1. Then \((X, d)\) and \((X, d')\) has the same topology. (Hint: Show that the identity maps \( \text{id}_X : (X, d) \to (X, d') \) and \( \text{id}_X : (X, d') \to (X, d) \) are continuous by either using \( \epsilon - \delta \)-method or showing that the pre-image of open sets are open.)

Exercise 0.5. Prove each of the following statements.

a) If \( Y \) is a subset of a topological space \( X \) with \( Y \subseteq F \subseteq X \) and \( F \) is closed then \( \bar{Y} \subseteq F \).
b) $Y$ is closed if and only if $Y = \bar{Y}$.

c) $\bar{\bar{Y}} = \bar{Y}$.

d) $A \cup B = \bar{A} \cup \bar{B}$.

e) $X \setminus \bar{Y} = X \setminus Y$.

f) $\bar{Y} = Y \cup \partial Y$ where $\partial Y = \bar{Y} \cap (X \setminus Y)$ (the boundary of $Y$).

g) $Y$ is closed if and only if $\partial Y \subseteq Y$.

h) $\partial Y = \emptyset$ if and only if $Y$ is both open and closed.

i) For $a < b \in \mathbb{R}$

$$\partial(a, b) = \partial[a, b] = \{a, b\}.$$  

**Exercise 0.6.** Show that

1) the subspace $(a, b)$ of $\mathbb{R}$ is homeomorphic to $\mathbb{R}$. (Hint: Use functions like $x \to \tan(\pi(cx + d))$ for suitable $c$ and $d$.)

2) the subspaces $(1, \infty), (0, 1)$ of $\mathbb{R}$ are homeomorphic. (Hint: $x \to 1/x$.)

3) $S^n \setminus \{(0, 0, \cdots, 0, 1)\}$ is homeomorphic to $\mathbb{R}^n$ with the usual topology. (Hint: Define $\phi: S^n \setminus \{(0, 0, \cdots, 0, 1)\} \to \mathbb{R}^n$ by

$$\phi(x_1, x_2, \cdots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \cdots, \frac{x_n}{1 - x_{n+1}}\right)$$

and $\psi: \mathbb{R}^n \to S^n \setminus \{(0, 0, \cdots, 0, 1)\}$ by

$$\psi(x_1, \cdots, x_n) = \frac{1}{1 + \|x\|^2}(2x_1, 2x_2, \cdots, 2x_n, \|x\|^2 - 1).$$

Harder Problems are given below.

**Problem 1.** For each $m, n \geq 0$, show that $S^{m+n}$ is homeomorphic to $S^m \wedge S^n$.


**Problem 2.** Show that $S^n/(\mathbb{Z}/2) \cong \mathbb{R}P^n$ and $\mathbb{R}P^n/\mathbb{R}P^{n-1} \cong S^n$.

**Hint:** You can do this problem by the following steps.

(1) Show that $\mathbb{R}P^n$ is Hausdorff. To prove this, let $l_1$ and $l_2$ be two elements in $\mathbb{R}P^n$, that is two lines in $\mathbb{R}^{n+1}$ passing the origin. Let $q: \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ be the quotient map and let $x, y \in \mathbb{R}^{n+1}$ with $\|x\| = \|y\| = 1$, $x \in l_1$ and $y \in l_2$.Let $\epsilon$ be a positive number such that $\epsilon < \min\{\|x + y\|, \|x - y\|\}$. (Such an $\epsilon$ exists because $x$ and $y$ are linearly independent vectors.) Consider the open balls $B_{\epsilon/2}(x)$ and $B_{\epsilon/2}(y)$. Show that $q^{-1}(q(B_{\epsilon/2}(x)))$ and $q^{-1}(q(B_{\epsilon/2}(y)))$ are disjoint open sets in $\mathbb{R}^{n+1} \setminus \{0\}$. By this, you get that $q(B_{\epsilon/2}(x))$ and $q(B_{\epsilon/2}(y))$ are disjoint open neighborhoods of $x$ and $y$, respectively. (So $\mathbb{R}P^n$ is Hausdorff by the definition.)
(2). Let $\pi: S^n \to \mathbb{R}P^n$ be the composite $S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$, that is $\pi(x)$ is the line passing $x$ and the origin. You find that $\pi(x) = \pi(-x)$. By using this, show that $\pi$ induces a well-defined function $\tilde{\pi}: S^n/(\mathbb{Z}/2) \to \mathbb{R}P^n$. Now you check that: 1) $\tilde{\pi}$ is a map (by the definition of quotient topology), 2) $\tilde{\pi}$ is onto and 3) $\tilde{\pi}$ is one-to-one. Now you show that $\tilde{\pi}$ is a homeomorphism as follows. Because $S^n$ is compact, the quotient $S^n/(\mathbb{Z}/2)$ is compact. Show the general statement that any map from a compact space to a Hausdorff space is a closed map [This is assertion 5 of Exercise 2.8.4. Say $f: X \to Y$ is such a map. Let $A$ be closed in the compact space $X$. Then $A$ is compact (Theorem 2.8.3) and so $f(A)$ is compact in the Hausdorff space $Y$ (Theorem 2.8.2). Thus $f(A)$ is closed (Theorem 2.8.5).] By this statement, $\tilde{\pi}$ is a closed bijective continuous function and so $\tilde{\pi}$ is a homeomorphism.

(3). Let $S^n_+ = \{x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}||x|| = 1$ and $x_{n+1} \geq 0\}$. Let $\tilde{\pi}: S^n_+ \to \mathbb{R}P^n$ be the restriction of the map $\pi$, that is, $\tilde{\pi}(x)$ is the line passing $x$ and the origin. Consider $S^{n-1}$ as a subspace of $S^n_+$ by

$$S^{n-1} = \{x = (x_1, \cdots, x_n, 0)||x|| = 1\}.$$  

Show that $\tilde{\pi}^{-1}(\mathbb{R}P^{n-1}) = S^{n-1}$ as a subspace of $S^n_+$. [Let $x \in S^n_+$. $\tilde{\pi}(x) \in \mathbb{R}P^{n-1}$ means that this line lies in the subspace $\mathbb{R}^n = \{(x_1, \cdots, x_n, 0)\} \subseteq \mathbb{R}^{n+1}$.] By pinching out $S^{n-1}$ and $\mathbb{R}P^{n-1}$ from $S^n_+$ and $\mathbb{R}P^n$, respectively, the map $\tilde{\pi}$ induces a bijective (continuous) map $\pi': S^n_+/S^{n-1} \to \mathbb{R}P^n/\mathbb{R}P^{n-1}$. Because $S^n_+/S^{n-1}$ is compact and $\mathbb{R}P^n/\mathbb{R}P^{n-1}$ is Hausdorff [Corollary 2.8.8], $\pi'$ is a homeomorphism. Now

$$S^n_+/S^{n-1} \cong D^n/S^{n-1} \cong S^n$$  

and so this finishes the proof.