Question 1(a). Since $\sum_{n=1}^{\infty} a_n$ is convergent, we have $\lim_{n \to \infty} a_n = 0$ and so there exists a positive integer $N$ such that $a_n = |a_n| = |a_n - 0| < 1$ for $n > N$. It follows
\[ a_n^2 = a_n \cdot a_n \leq 1 \cdot a_n = a_n \]
for $n > N$. By the comparison test, the positive series $\sum_{n=1}^{\infty} a_n^2$ is convergent. □

Question 1 (b). Let $a_n = \frac{1}{n^2}$. Then $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} \sqrt{a_n}$ is divergent by the $p$-series. □

Question 2. Let $f(x) = \frac{1}{x(1 + \ln x)}$. Then $f(x)$ is a positive, continuous monotone decreasing function over $[1, +\infty)$. Observe that
\[
\int f(x)dx = \int \frac{1}{x(1 + \ln x)} \, dx = \int \frac{1}{1 + y} \, dy = \ln(1 + y) + C = \ln(1 + \ln x) + C,
\]
where $y = \ln x$. The integral $\int_{1}^{\infty} f(x) \, dx$ is divergent and so is the series $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)}$ by the integral test. □

Question 3 (a). Let $a_n = \frac{(3n)!}{6^n n!(2n)!}$. Then
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{[3(n + 1)]!6^n n!(2n)!}{6^{n+1}(n + 1)!(2n + 1)!(3n)!} = \lim_{n \to \infty} \frac{(3n + 3)(3n + 2)(3n + 1)}{6(n + 1)(2n + 2)(2n + 1)} = \frac{3 \cdot 3 \cdot 3}{6 \cdot 2 \cdot 2} = \frac{27}{24} > 1.
\]
Thus the series $\sum_{n=1}^{\infty} \frac{(3n)!}{6^n n!(2n)!}$ is divergent by the ratio test. □

Question 3 (b). Let $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} 2 \left(1 - \frac{1}{n}\right)^n = \frac{2}{e} < 1$, the series $\sum_{n=1}^{\infty} a_n$ is convergent by the ratio test. □
Question 4 (a). Let $a_n = \frac{5n^2 \cdot 3^n}{4^{n+4}}$. Then

$$
\lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \frac{5\frac{2}{n}(\sqrt{n})^2 \cdot 3}{4 \cdot 4^n} = \frac{1 \cdot 1^2 \cdot 3}{4 \cdot 1} = \frac{3}{4} < 1.
$$

Thus the series $\sum_{n=1}^{\infty} \frac{5n^2 \cdot 3^n}{4^{n+4}}$ is convergent by the simplified root test. \(\square\)

Question 4 (b). Let $a_n = \frac{3^{2n}}{5^n} \left(1 - \frac{1}{2n}\right)^{n^2}$. Then

$$
\lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \frac{3^2}{5} \left(1 + \frac{-\frac{1}{2}}{n}\right)^n = \frac{9}{5}e^{-\frac{1}{2}} = \frac{9}{5\sqrt{e}} > 1
$$

because $e < \frac{9^2}{3^2} = \frac{81}{25} = 3.24$. Thus the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{5^n} \left(1 - \frac{1}{2n}\right)^{n^2}$ is divergent by the simplified root test. \(\square\)

Question 4 (c). Let $a_n$ be the $n$-term in the series. Then $a_{2n-1} = \frac{1}{4^{2n-1}}$ and $a_{2n} = \frac{1}{4^{2n}}$. Thus

$$
\sqrt[n]{a_n} = \left\{ \begin{array}{ll}
\frac{1}{4} & \text{if } n \text{ is odd} \\
\frac{1}{5} & \text{if } n \text{ is even}
\end{array} \right.
$$

and so $\limsup_{n \to \infty} \sqrt[n]{a_n} = \frac{1}{4} < 1$. Hence the series is convergent by the root test. \(\square\)