INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of TWO (2) sections: Section A and Section B. It contains a total of SEVEN (7) questions and comprises FOUR (4) printed pages.

2. Answer ALL questions in Section A. Section A carries a total of 60 marks.

3. Answer no more than TWO (2) questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.
SECTION A

Answer all the questions in this section. Section A carries a total of 60 marks.

Question 1 [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) \( \sqrt[3]{\frac{8n + n^8 + n!}{n! + (\ln n)^8}} \)

(b) \( \frac{\sqrt{n}}{\sqrt{2n + 1} - \sqrt{n}} \)

(c) \( \left\{ \frac{2n}{2n - 1} \right\}^{3n} \)

(d) \( \left\{ 3n \sin \left( \frac{1}{n} \right) + (5^n + 3^n)^{\frac{1}{n}} \right\} \)

Question 2 [16 marks]

Determine the absolute convergence, conditional convergence or divergence of each of the following series. Justify your answers.

(a) \( \sum_{n=1}^{\infty} \frac{1}{n + \ln n} \)

(b) \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{2\sqrt{n} + 1} \)

(c) \( \sum_{n=1}^{\infty} (-1)^n \frac{1 + \sin n}{n(1 + \ln n)^2} \)

(d) \( \sum_{n=1}^{\infty} \frac{2^{4n}(n!)^3}{(3n)!} \)
Question 3 [12 marks]
Find the interval of convergence of the following power series.

(a) \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n^2 - \ln n} \]

(b) \[ \sum_{n=1}^{\infty} \frac{(2x - 1)^n}{\sqrt{n}} \]

Question 4 [16 marks]

(a) Find the general solutions of the following differential equations.

(i) \[ y' - x^2 y = y \]

(ii) \[ y'' + 5y' + 6y = 0 \]

(b) Evaluate the limit

\[ \lim_{t \to 0} \frac{1 - \cos(t^5)}{t \sin(t^3) - t^4} \]

SECTION B
Answer not more than TWO (2) questions from this section. Each question in this section carries 20 marks.

Question 5 [20 marks]

(a) Evaluate \[ \lim_{n \to \infty} \int_{0}^{1} \left( \frac{x^2 + x + 1}{4} \right)^n \sin nx dx \]. Justify your answer.

(b) Find the first three terms (up to and including the quadratic term) for the Taylor series for the function \( f(x) = \sqrt{x} \) expanded around \( x = 8 \).

(c) Use an appropriate series to find a decimal approximation to \( \sqrt[3]{8.1} \) that is accurate to within \( 10^{-6} \). Justify your answer.
Question 6 [20 marks]

(a) Solve the following initial value problem:
\[ y'' + 2y' + y = te^{-2t}; \quad y(0) = 0 \quad y'(0) = 2. \]

(b) Consider the sequence \( \{a_n\} \) given by
\[ a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2} + a_n, \quad \text{for} \quad n \geq 1. \]
Show that \( \{a_n\} \) converges, and find its limit.

(c) Does the series of functions \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^x} \) converge uniformly on the interval \((0, +\infty)\)? Justify your answer.

Question 7 [20 marks]

(a) Let \( \{a_n\} \) be a bounded sequence of real numbers. Show that
\[ \limsup_{n \to \infty} \sqrt{|a_n|} = \sqrt{\limsup_{n \to \infty} |a_n|}. \]

(b) (i) Let \( a > 1 \) be any positive number greater than 1. Show that the series of functions \( \sum_{n=1}^{\infty} \frac{\ln n}{n^x} \) converges uniformly over the interval \([a, +\infty)\).

(ii) Using (i) or otherwise, show that the function \( \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \) is differentiable on \((1, +\infty)\).