Take-home Exam 2

*In order to get the full marks for the project of take-home exams, you need to have 50 or more points. Well there are 10 points in this take-home exam.*

**Question 1** [2 points, 1 for each part]

Let \( a_1 \) and \( b_1 \) be positive numbers with \( a_1 > b_1 \). Let \( a_2 = \frac{a_1 + b_1}{2} \) be their arithmetic mean and let \( b_2 = \sqrt{a_1 b_1} \) be their geometric mean. Repeat this process so that, in general,

\[
a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}.
\]

(a) Show by mathematical induction that

\( a_n > a_{n+1} > b_{n+1} > b_n \).

(b) Prove that \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n \).

*(Note. Gauss called this the common value of these limits the *arithmetic-geometric mean* of the numbers \( a = a_1 \) and \( b = b_1 \).)*

**Question 2.** [3 points, 1 for each part]

Find limit inferior and limit superior of each of the following sequences.

(a) \( \left\{ \frac{n + (-1)^n n^2}{n^2 + 1} \right\} \).

(b) \( \{1.5 + (-1)^n \} \).

(c) \( \left\{ \left( 1 + \frac{1}{n} \right) \left( 1 + \sin \frac{n\pi}{8} \right)^\frac{1}{n} \right\} \).

**Question 3** [5 points, 1 for each part]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) \( \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2 + 2k - 1} \).

(b) \( \sum_{n=1}^{\infty} \frac{1}{n(2 + \ln n)} \).

(c) \( \sum_{n=1}^{\infty} 6^n \left( 1 - \frac{2}{n + 1} \right)^n \).

(d) \( \sum_{n=1}^{\infty} \frac{n^n}{3^n \cdot n!} \).

(e) \( \sum_{k=1}^{\infty} \frac{\sqrt{k + 1} - \sqrt{k}}{k} \).