Take-home Exam 4

In order to get the full marks for the project of take-home exams, you need to have 50 or more points. Well there are 10 points in this take-home exam.

Question 1 [4 points, 1 for each part]
For each of the following sequence of functions, determine whether it converges pointwise to a function, and find the limiting function if it exists. Justify your answers.

(a) \( \left\{ \left( 1 - \frac{x^2}{n} \right)^{nx} \right\}, \ x \in \mathbb{R}. \)

(b) \( \left\{ (\cos x)^{2n} \right\}, \ x \in \mathbb{R}. \)

(c) \( \left\{ \frac{\sin nx}{\cos nx + nx} \right\}, \ x \in [1, +\infty). \)

(d) \( \{f_n(x)\}, \ f_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1 + x^2)^k}, \ x \in \mathbb{R}. \)

Question 2. [6 points, 1 for each part] Determine whether the following sequences of functions converge uniformly on the indicated intervals. Justify your answers.

(a) \( F_n(x) = \frac{x^n}{1 + x^n}, \ x \in [0, \frac{1}{2}]. \)

(b) \( F_n(x) = \frac{x^n}{1 + x^n}, \ x \in [0, 1]. \)
   (Hint: Find the limiting function \( F(x) \) and check that the limiting function is not continuous but each \( F_n \) is, and from this conclude that the sequence of functions does not converge uniformly.)

(c) \( F_n(x) = x + \frac{x}{n} \sin nx, \ x \in [-a, a], \ a > 0. \)

(d) \( F_n(x) = x + \frac{\pi}{n} \sin nx, \ x \in \mathbb{R}. \)
   (Hint: Try to find a lower bound of \( T_n = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \) by taking \( x = \frac{2n\pi}{n} \).

(e) \( F_n(x) = \frac{n^2 x^3 \sin nx}{1 + x^n}, \ x \in [0, \frac{1}{2}]. \)

(f) \( F_n(x) = nx \left( 1 - x^2 \right)^n, \ x \in [0, 1]. \)