1. Examples of Manifolds

1.1. Open Stiefel Manifolds and Grassmann Manifolds. The open Stiefel manifold is the space of \( k \)-tuples of linearly independent vectors in \( \mathbb{R}^n \):

\[
\tilde{V}_{k,n} = \{(\tilde{v}_1, \ldots, \tilde{v}_k) \mid \tilde{v}_i \in \mathbb{R}^n, \{\tilde{v}_1, \ldots, \tilde{v}_k\} \text{ linearly independent}\},
\]

where \( \tilde{V}_{k,n} \) is considered as the subspace of \( k \times n \) matrixes \( M(k, n) \cong \mathbb{R}^{kn} \). Since \( \tilde{V}_{k,n} \) is an open subset of \( M(k, n) = \mathbb{R}^{kn} \), \( \tilde{V}_{k,n} \) is an open submanifold of \( \mathbb{R}^{kn} \).

The Grassmann manifold \( G_{k,n} \) is the set of \( k \)-dimensional subspaces of \( \mathbb{R}^n \), that is, all \( k \)-planes through the origin. Let

\[
\pi : \tilde{V}_{k,n} \rightarrow G_{k,n}
\]

be the quotient by sending \( k \)-tuples of linearly independent vectors to the \( k \)-planes spanned by \( k \) vectors. The topology in \( G_{k,n} \) is given by quotient topology of \( \pi \), namely, \( U \) is an open set of \( G_{k,n} \) if and only if \( \pi^{-1}(U) \) is open in \( \tilde{V}_{k,n} \).

For \( (\tilde{v}_1, \ldots, \tilde{v}_k) \in \tilde{V}_{k,n} \), write \( (\tilde{v}_1, \ldots, \tilde{v}_k) \) for the \( k \)-plane spanned by \( \tilde{v}_1, \ldots, \tilde{v}_k \). Observe that two \( k \)-tuples \( (\tilde{v}_1, \ldots, \tilde{v}_k) \) and \( (\tilde{w}_1, \ldots, \tilde{w}_k) \) span the same \( k \)-plane if and only if each of them is basis for the common plane, if and only if there is nonsingular \( k \times k \) matrix \( P \) such that

\[
P(\tilde{v}_1, \ldots, \tilde{v}_k)^T = (\tilde{w}_1, \ldots, \tilde{w}_k)^T.
\]

This gives the identification rule for the Grassmann manifold \( G_{k,n} \). Let \( \text{GL}_k(\mathbb{R}) \) be the space of general linear groups on \( \mathbb{R}^k \), that is, \( \text{GL}_k(\mathbb{R}) \) consists of \( k \times k \) nonsingular matrixes, which is an open subset of \( M(k, k) = \mathbb{R}^{k^2} \). Then \( G_{k,n} \) is the quotient of \( \tilde{V}_{k,n} \) by the action of \( \text{GL}_k(\mathbb{R}) \).

First we prove that \( G_{k,n} \) is Hausdorff. If \( k = n \), then \( G_{k,n} \) is only one point. So we assume that \( k < n \). Given an \( k \)-plane \( X \) and \( \bar{w} \in \mathbb{R}^n \), let \( \rho_{\bar{w}} \) be the square of the Euclidian distance from \( \bar{w} \) to \( X \). Let \( \{e_1, \ldots, e_k\} \) be the orthogonal basis for \( X \), then

\[
\rho_{\bar{w}}(X) = \bar{w} \cdot \bar{w} - \sum_{j=1}^{k} (\bar{w} \cdot e_j)^2.
\]

Fixing any \( \bar{w} \in \mathbb{R}^n \), we obtain the continuous map

\[
\rho_{\bar{w}} : G_{k,n} \rightarrow \mathbb{R}
\]

because \( \rho_{\bar{w}} \circ \pi : \tilde{V}_{k,n} \rightarrow \mathbb{R} \) is continuous and \( G_{k,n} \) given by the quotient topology. (Here we use the property of quotient topology that any function \( f \) from the quotient space \( G_{k,n} \) to any space is continuous if and only if \( f \circ \pi \) from \( \tilde{V}_{k,n} \) to that space is continuous.) Given any two distinct points \( X \) and \( Y \) in \( G_{k,n} \), we can choose a \( \bar{w} \) such that \( \rho_{\bar{w}}(X) \neq \rho_{\bar{w}}(Y) \). Let \( V_1 \) and \( V_2 \) be disjoint open subsets of \( \mathbb{R} \) such that \( \rho_{\bar{w}}(X) \in V_1 \) and \( \rho_{\bar{w}}(Y) \in V_2 \). Then \( \rho_{\bar{w}}^{-1}(V_1) \) and \( \rho_{\bar{w}}^{-1}(V_2) \) are two open subset of \( G_{k,n} \) that separate \( X \) and \( Y \), and so \( G_{k,n} \) is Hausdorff.

Now we check that \( G_{k,n} \) is manifold of dimension \( k(n - k) \) by showing that, for any \( X \) in \( G_{k,n} \), there is an open neighborhood \( U_X \) of \( \alpha \) such that \( U_X \cong \mathbb{R}^{k(n-k)} \).
Let \( X \in G_{k,n} \) be spanned by \((\vec{v}_1, \ldots, \vec{v}_k)^T\). There exists a nonsingular \( n \times n \) matrix \( Q \) such that
\[
(\vec{v}_1, \ldots, \vec{v}_k)^T = (I_k, 0)Q,
\]
where \( I_k \) is the unit \( k \times k \)-matrix. Fixing \( Q \), define
\[
X_n = \{(P_k, B_{k,n-k})Q \mid \det(P_k) \neq 0, B_{k,n-k} \in M(k, n-k)\} \subseteq \tilde{V}_{k,n}.
\]
Then \( E_X \) is an open subset of \( \tilde{V}_{k,n} \). Let \( U_X = \pi(E_X) \subseteq G_{k,n} \). Since \( \pi^{-1}(U_X) = E_X \) is open in \( \tilde{V}_{k,n} \), \( U_X \) is open in \( G_{k,n} \) with \( X \in U_X \).

From the commutative diagram
\[
\begin{array}{ccc}
\text{GL}_k(\mathbb{R}) \times M(k, n-k) & \overset{(P, A) \mapsto (P, PA)Q}{\cong} & E_X \\
\downarrow \text{proj.} & & \downarrow \pi \\
M(k, n-k) & \overset{A \mapsto \langle(I_k, A)Q\rangle}{\cong} & U_X,
\end{array}
\]
\( U_X \) is homeomorphic to \( M(k, n-k) = \mathbb{R}^{k(n-k)} \) and so \( G_{k,n} \) is a (topological) manifold.

For checking that \( G_{k,n} \) is a smooth manifold, let \( X \) and \( Y \in G_{k,n} \) be spanned by \((\vec{v}_1, \ldots, \vec{v}_k)^T\) and \((\vec{w}_1, \ldots, \vec{w}_k)^T\), respectively. There exists nonsingular \( n \times n \) matrixes \( Q \) and \( \tilde{Q} \) such that
\[
(\vec{v}_1, \ldots, \vec{v}_k)^T = (I_k, 0)Q, \quad (\vec{w}_1, \ldots, \vec{w}_k)^T = (I_k, 0)\tilde{Q}.
\]
Consider the maps:
\[
M(k, n-k) \overset{\phi_X^{-1}}{\longrightarrow} U_X \quad A \mapsto \langle(I_k, A)Q\rangle
\]
\[
M(k, n-k) \overset{\phi_Y^{-1}}{\longrightarrow} U_Y \quad A \mapsto \langle(I_k, A)\tilde{Q}\rangle.
\]
If \( Z \in U_X \cap U_Y \), then
\[
Z = \langle(I_k, A_Z)Q\rangle = \langle(I_k, B_Z)\tilde{Q}\rangle
\]
for unique \( A, B \in M(k, n-k) \). It follows that there is a nonsingular \( k \times k \) matrix \( P \) such that
\[
(I_k, B_Z)\tilde{Q} = P(I_k, A_Z)Q \iff (I_k, B_Z) = P(I_k, A_Z)Q\tilde{Q}^{-1}.
\]
Let
\[
T = Q\tilde{Q}^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.
\]
Then
\[
(I_k, B_Z) = (P, PA_Z)T = (PT_{11} + PA_ZT_{21}, PT_{12} + PA_ZT_{22})
\]
\[
\begin{cases}
I_k = P(T_{11} + A_ZT_{21}) \\
B_Z = P(T_{12} + A_ZT_{22}).
\end{cases}
\]
It follows that
\( Z \in U_X \cap U_Y \) if and only if \( \det(T_{11} + AZT_{21}) \neq 0 \) (that is, \( T_{11} + AZT_{21} \) is invertible).

From the above, the composite
\[
\phi_X(U_X \cap U_Y) \xrightarrow{\phi_X^{-1}} U_X \cap U_Y \xrightarrow{\phi_Y} M(k, n)
\]
is given by
\[
A \mapsto (T_{11} + AT_{21})^{-1}(T_{12} + AT_{22}),
\]
which is smooth. Thus \( G_{k,n} \) is a smooth manifold.

As a special case, \( G_{1,n} \) is the space of lines (through the origin) of \( \mathbb{R}^n \), which is also called **projective space** denoted by \( \mathbb{RP}^{n-1} \). From the above, \( \mathbb{RP}^{n-1} \) is a manifold of dimension \( n - 1 \).

### 1.2. Stiefel Manifold.

The **Stiefel manifold**, denoted by \( V_{k,n} \), is defined to be the set of \( k \) orthogonal unit vectors in \( \mathbb{R}^n \) with topology given as a subspace of \( \tilde{V}_{k,n} \subseteq M(k,n) \). Thus
\[
V_{k,n} = \{ A \in M(k,n) \mid A \cdot A^T = I_k \}.
\]
We prove that \( V_{k,n} \) is a smooth submanifold of \( M(k,n) \) by using Pre-image Theorem.

Let \( S(k) \) be the space of symmetric matrixes. Then \( S(k) \cong \mathbb{R}^{\frac{(k+1)k}{2}} \) is a smooth manifold of dimension. Consider the map
\[
f : M(k,n) \to S(k) \quad A \mapsto AA^T.
\]
For any \( A \in M(k,n) \), \( T_{fA} : T_A(M(k,n)) \to T_{f(A)}(S(k)) \) is given by setting \( T_{fA}(B) \) is the directional derivative along \( B \) for any \( B \in T_A(M(k,n)) \), that is,
\[
T_{fA}(B) = \lim_{s \to 0} \frac{f(A + sB) - f(A)}{s}
\]
\[
= \lim_{s \to 0} \frac{(A + sB)(A + sB)^T - AA^T}{s}
\]
\[
= \lim_{s \to 0} \frac{AA^T + sAB^T + sBA^T + s^2BB^T - AA^T}{s}
\]
\[
= AB^T + BA^T.
\]
We check that \( T_{fA} : T_A(M(k,n)) \to T_{f(A)}(S(k)) \) is surjective for any \( A \in f^{-1}(I_k) \).

By the identification of \( M(k,n) \) and \( S(k) \) with Euclidian spaces, \( T_A(M(k,n)) = M(k,n) \) and \( T_{f(A)}(S(k)) = S(k) \). Let \( A \in f^{-1}(I_k) \) and let \( C \in T_{f(A)}(S(k)) \). Define
\[
B = \frac{1}{2}CA \in T_A(M(k,n)).
\]
Then
\[
Tf_A(B) = AB^T + BA^T = \frac{1}{2}AA^T + \frac{1}{2}CAA^T \xrightarrow{AA^T = I_k} \frac{1}{2}C + \frac{1}{2}C = CT = C.
\]
Thus \( T : T_A(M(k,n)) \to T_{f(A)}(S(k)) \) is onto and so \( I_k \) is a regular value of \( f \). Thus, by Pre-image Theorem, \( V_{k,n} = f^{-1}(I_k) \) is a smooth submanifold of \( M(k,n) \) of dimension
\[
k\frac{n}{2} - \frac{(k+1)k}{2} = \frac{k(2n-k-1)}{2}.
\]
**Special Cases:** When \( k = n \), then \( V_{n,n} = O(n) \) the orthogonal group. From the above, \( O(n) \) is a (smooth) manifold of dimension \( \frac{n(n-1)}{2} \). **Note:** \( O(n) \) is a Lie group,
namely, a smooth manifold plus a topological group such that the multiplication and inverse are smooth.)

When \( k = 1 \), then \( V_{1,n} = S^{n-1} \) which is manifold of dimension \( n - 1 \).

When \( k = n - 1 \), then \( V_{n-1,n} \) is a manifold of dimension \( \frac{(n-1)n}{2} \). One can check that

\[
V_{n-1,n} \cong SO(n)
\]

the subgroup of \( O(n) \) with determinant 1. In general case, \( V_{k,n} = O(n)/O(n - k) \).

As a space, \( V_{k,n} \) is compact. This follows from that \( V_{k,n} \) is a closed subspace of the \( k \)-fold Cartesian product of \( S^{n-1} \) because \( V_{k,n} \) is given by \( k \) unit vectors \( (\vec{v}_1, \ldots, \vec{v}_k)^T \) in \( \mathbb{R}^n \) that are solutions to \( \vec{v}_i \cdot \vec{v}_j = 0 \) for \( i \neq j \), and the fact that any closed subspace of compact Hausdorff space is compact. The composite

\[
V_{k,n} \hookrightarrow \tilde{V}_{k,n} \xrightarrow{\pi} G_{k,n}
\]

is onto and so the Grassmann manifold \( G_{k,n} \) is also compact. Moreover the above composite is a smooth map because \( \pi \) is smooth and \( V_{k,n} \) is a submanifold. This gives the diagram

\[
\begin{array}{cccc}
V_{k,n} & \hookrightarrow & \tilde{V}_{k,n} & \xrightarrow{\pi} G_{k,n} \\
\downarrow \text{submanifold} & & \downarrow \text{submersion at } I_k & \\
M(k, n) & \xrightarrow{\text{submersion at } I_k} & S(k) & \\
\downarrow \text{smooth} & & & \\
G_{k,n} & & & \\
\end{array}
\]

**Note.** By the construction, \( G_{k,n} \) is the quotient of \( V_{k,n} \) by the action of \( O(k) \). This gives identifications:

\[
G_{k,n} = V_{k,n}/O(k) = O(n)/(O(k) \times O(n - k)).
\]