Question 1. Let $M$ and $N$ be smooth manifolds.

1) Show that $T_{(x,y)}(M \times N) = T_x(M) \times T_y(N)$.

2) Let $p: M \times N \to M$ be the projection map $(x, y) \mapsto x$. Prove that $Tp: T_x(M) \times T_y(N) \to T_x(M)$ is the analogous projection $(\vec{v}, \vec{w}) \mapsto \vec{v}$.

3) Fixing any $y \in N$, let $j: M \to M \times N$ be the inclusion $x \mapsto (x, y)$. Show that $Tj(\vec{v}) = (\vec{v}, 0)$.

4) Let $f: M \to M'$ and $g: N \to N'$ be smooth maps. Prove that $T(f \times g)(x,y) = Tfx \times Tgy$.

Question 2. Let $M$ be a smooth manifold.

1) Let $\Delta: M \to M \times M$ be the diagonal map $x \mapsto (x, x)$. Prove that $T\Delta_x(\vec{v}) = (\vec{v}, \vec{v})$.

2) Let $\Delta(M) = \{(x, x) \mid x \in M\} \subseteq M \times M$ be the diagonal. Show that the tangent space $T_{(x,x)}(\Delta(M))$ is the diagonal of $T_x(M) \times T_x(M)$.

Question 3. Prove the following statements:

1) If $f$ and $g$ are immersions, then so is $f \times g$.

2) If $f$ and $g$ are immersions, then so is $g \circ f$.

3) If $f$ is an immersion, then so is $f$ restricted to any submanifold of its domain.

4) If $\dim M = \dim N$, then immersions $f: M \to N$ are the same as local diffeomorphisms.

Question 4. Check the map

$$\mathbb{R}^1 \to \mathbb{R}^2, \quad t \mapsto \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}\right)$$

is an embedding. Prove that its image is one nappe of the hyperbola $x^2 - y^2 = 1$.

Question 5. The smooth links can be regarded as 1-dimensional submanifolds of $\mathbb{R}^3$. The links can be also regarded as embeddings of disjoint union of finite copies of $S^1$ into $\mathbb{R}^3$. Draw a nontrivial links consisting of 3 components with the property that it becomes a trivial link after removing any one of the links. [This kind of links is called Brunnian links.]

Let $f: M \to N$ be a smooth map. A point $y \in N$ is called a critical value if $Tf: T_x(M) \to T_y(N)$ is not surjective for some $x \in f^{-1}(y)$. (Namely, if $y$ is not regular.)

Question 6. Check that 0 is the only critical value of the map $f: \mathbb{R}^3 \to \mathbb{R}^1$ defined by

$$f(x, y, z) = x^2 + y^2 - z^2.$$
Prove that if \( a \) and \( b \) are either both positive or both negative, then \( f^{-1}(a) \) and \( f^{-1}(b) \) are diffeomorphic. [Hint: Consider scalar multiplication by \( \sqrt{b/a} \) on \( \mathbb{R}^3 \).] Pictorially examine the catastrophic change in the topology of \( f^{-1}(c) \) as \( c \) passes through the critical value.

**Problem 7.** Let \( M \) and \( Z \) be transversal submanifolds of \( N \). Prove that if \( y \in M \cap Z \), then
\[
T_y(M \cap Z) = T_y(M) \cap T_y(Z).
\]

**Problem 8.** For which values of \( a \) does the hyperboloid defined by \( x^2 + y^2 - z^2 = 1 \) intersect the sphere \( x^2 + y^2 + z^2 = a \) transversally? What does the intersection look like for different values of \( a \)?