Homework 4

**Question 1.** Prove that a \( k \)-dimensional vector bundle \( \xi \) is trivial if and only if it has \( k \) cross-sections \( s_1, \ldots, s_k \) such that each \( s_1(b), \ldots, s_k(b) \) are linearly independent for each \( b \in B \).

**Question 2.** Let \( \xi \) and \( \eta \) be vector bundles over \( B \) and let \( f \) be a cross-section of the bundle \( \text{Hom}(\xi, \eta) \). If the rank of the linear transformation \( f(b): F_b(\xi) \to F_b(\eta) \) is locally constant as a function of \( b \), define the kernel \( \text{Ker} f \subseteq \xi \) and the cokernel \( \text{Coker} f \), and prove that they are locally trivial.

**Question 3.** Let \( B \) be a compact Hausdorff space and let \( C^0(B) \) be the ring of continuous real valued functions on \( B \). For any vector bundle \( \xi \) over \( B \) let \( \Gamma(\xi) \) denote the \( C^0(B) \)-module consisting of all cross-sections of \( \xi \).
   a) Show that \( \Gamma(\xi \oplus \eta) \cong \Gamma(\xi) \oplus \Gamma(\eta) \).
   b) Show that \( \xi \cong \eta \) if and only if \( \Gamma(\xi) \cong \Gamma(\eta) \) as \( C^0(B) \)-modules.
   c) Show that \( \xi \) is trivial if and only if \( \Gamma(\xi) \) is a free \( C^0(B) \)-bundle.

[Hint: These are some statements for special cases from the paper: R. Swan, *vector bundles and projective modules*, Trans. Amer. Math. Soc., 105 (1962), 264-277.]

**Question 4.** Let \( \phi: V \to W \) be linear isomorphism of finite dimensional vector spaces. Show that the matrix of \( (\phi^{-1})^*: W^* \to V^* \) is the transpose of the inverse of the matrix of \( \phi \). [Note. You may use the formula \( \langle \phi^*x, y \rangle = \langle x, \phi y \rangle \).] Recall that a Riemann metric is a tensor field \( g \in T^0_2(M) \) such that for each \( m \), \( g_m \) is an inner product, that is, positive definite symmetric and bilinear.

**Question 5.** Determine Riemann metrics \( g \in T^0_2(\mathbb{R}^2) = (\mathbb{R}^2)^* \otimes (\mathbb{R}^2)^* \).

**Question 6.** Show that
\[ \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \] \[ \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(n,m)\mathbb{Z}, \] where \( (n,m) \) is the greatest common factor of \( m \) and \( n \).