ON FUNCTIONAL DECOMPOSITIONS OF SELF-SMASH PRODUCTS

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Abstract. We give a decomposition formula for n-fold self smash of a two-cell suspension X localized at 2, in which the mod 2 homology of each factor in the decomposition is explicitly given and is indecomposable over the Steenrod algebra if X is a suspension of RP2, CP2, HP2 or KP2. The method has consequences in the modular representation theory of the symmetric group where it leads to a computation of the submatrix for the decomposition matrix of the group algebra \( \mathbb{Z}/2[S_n] \) which correspond to partitions of length 2. In particular this yields a derivation of the explicit formula due to Erdmann which gives the multiplicities in the decomposition of \( \mathbb{Z}/2[S_n] \) of the indecomposable projective modules which correspond to those partitions.

1. Introduction

Let \( X = \Sigma Y \) be the suspension of a p-local CW-complex Y and let \( X^{(n)} \) be the n-fold self smash product of X. In this paper, we study functorial decompositions of \( X^{(n)} \) using the canonical representation of symmetric group \( S_n \). If \( V = H_n(X; \mathbb{Z}/p) \) then any such decomposition gives a \( \mathbb{Z}/p[S_n] \)-module decomposition of \( V^\otimes n \), and to an extent this process is reversible. Such decompositions also commute with the action of the Steenrod algebra on \( \tilde{H}^*(X; \mathbb{Z}/p) \) or by taking hom-duals, the action of the opposite of the Steenrod algebra on V. We shall use homology and simply say “Steenrod algebra” for the opposite algebra of the Steenrod algebra.

The representation of general linear groups and symmetric groups on tensor products has been studied by various people (for example [1, 9, 10, 11, 17]), where the use of hyperalgebras play an important role. There are certain connections between the hyperalgebras and the Steenrod algebra (See Section 3). Topologists tend to use the Steenrod algebra rather than the hyperalgebras. (See for example work of Gray, Kuhn, Mitchell, or Smith.)

We turn our attention to the case where \( p = 2 \). In studying Steenrod module decompositions of the homology, we find that for in the case where X is a 2-cell complex, there is a special property of the 3-fold self tensor (Lemma 4.3.) From this, we find a recursive algorithm to produce an explicit decomposition of n-fold self tensor (Proposition 4.4. Theorem 4.6 says that this decomposition is in fact a complete
decomposition over the Steenrod algebra. By using general results in representation theory, Theorem 5.8 concludes that this decomposition is a complete functorial decomposition (See section 2 for the definition of functorial decompositions).

We are indebted to Prof. James for pointing out to us the prior work of Erdmann ([7]), who used the methods of representation theory and quasi-hereditary algebras, and in particular the work of Donkin ([6]) to derive the algebraic consequences of our geometric decomposition. Donkins adaptation of Ringel’s work ([14] on quasi-heritary algebras led to a decomposition of “tilting modules” which contained a decomposition analogous to that of Lemma 4.3 which was exploited by Erdmann. Neither the algebraic nor the geometric approach have yet had much success in generalizing these results to complexes consisting of 3 or more cells.

We start with some observations. Let $S_n$ act on $X^{(n)}$ by permuting factors. Then we obtain a representation $\theta: S_n \to [X^{(n)}, X^{(n)}]$. For $n \geq 2$, the function $\theta$ extends to a $\mathbb{Z}$-linear map $\mathbb{Z}(\theta): \mathbb{Z}(S_n) \to [X^{(n)}, X^{(n)}]$ because the co-H-space $X^{(n)}$ is homotopy co-associative and co-commutative and each map $\sigma: X^{(n)} \to X^{(n)}$ is an co-H-map for $\sigma \in S_n$. Since $X$ is a $p$-local suspension, the $\mathbb{Z}$-linear map $\mathbb{Z}(\theta)$ extends to a $\mathbb{Z}(p)$-linear map $\mathbb{Z}(p)(\theta): \mathbb{Z}(p)(S_n) \to [X^{(n)}, X^{(n)}]$, where $\mathbb{Z}(p)$ is the ring of $p$-local integers. This gives a homotopy representation of $\mathbb{Z}(p)(S_n)$ on $X^{(n)}$. Now let

$$1 = \sum_\alpha e_\alpha$$

be an orthogonal decomposition of the identity in $\mathbb{Z}(p)(S_n)$ in terms of primitive idempotents. Let $f_\alpha: X^{(n)} \to X^{(n)}$ be a map such that the homotopy class $[f_\alpha] = \mathbb{Z}(p)(e_\alpha)$. Then we obtain a homotopy orthogonal decomposition of the identity map $\text{id}_{X^{(n)}}: X^{(n)} \to X^{(n)}$

$$\text{id}_{X^{(n)}} \simeq \sum_\alpha f_\alpha.$$ 

Let $e_\alpha(Y)$ be the homotopy colimit

$$e_\alpha(Y) = \text{hocolim}_{f_\alpha} X^{(n)}$$

for each $\alpha$. Then $e_\alpha(Y)$ is a homotopy retract of $X^{(n)}$ because $f_\alpha$ is a homotopy idempotent. Since $f_\alpha \circ f_\beta$ is null homotopic for $\alpha \neq \beta$, one gets a homotopy decomposition

$$X^{(n)} = (\Sigma Y)^{(n)} \simeq \bigvee_\alpha e_\alpha(Y).$$

Furthermore, this decomposition is functorial with respect to $Y$. In other words, each $e_\alpha$ is a functor from spaces to co-H-spaces. (Note: for each $\alpha$, we can fix a choice of
the representative $f_\alpha$. This gives a particular choice of space $e_\alpha(Y)$ such that $e_\alpha(Y)$ is strictly functorial with respect to $Y$. Furthermore, there is a choice of map
\[ \phi: X^{(n)} = (\Sigma Y)^{(n)} \to \bigvee_\alpha e_\alpha(Y) \]
such that $\phi$ is strictly functorial on $Y$ and is a homotopy equivalence for each $Y$.) Recall that up to conjugation primitive idempotents in $\mathbb{Z}_p(S_n)$ are in one-to-one correspondence with $p$-regular partitions of $n$. A (proper) partition of $n$ is a sequence of positive integers $\lambda = (\lambda_1, \cdots, \lambda_s)$ such that
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \text{ and } \sum_{j=1}^s \lambda_j = n. \]
$s$ is called the length of $\lambda$, and denoted by $\text{Len}(\lambda)$. A partition $\lambda = (\lambda_1, \cdots, \lambda_s)$ is called $p$-regular if there is no subscript $i$ with $1 \leq i \leq s$ such that $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p-1}$. For each $p$-regular partition $\lambda$, there is an idempotent $Q_\lambda \in \mathbb{Z}_p(S_n)$ such that the (left) ideal generated by $Q_\lambda$ is a projective cover of the Specht module corresponding to $\lambda$ (See [11, 2, 3] for details). Any primitive idempotent in $\mathbb{Z}_p(S_n)$ is conjugate to one and only one $Q_\lambda$. Let $d_\lambda$ be the number of idempotents in decomposition 1 which are conjugate to $Q_\lambda$. We call $d_\lambda$ the multiplicity of primitive idempotent $Q_\lambda$ in $\mathbb{Z}_p(S_n)$. Let $Q_\lambda(Y)$ be the homotopy colimit
\[ Q_\lambda(Y) = \text{hocolim}_{e^{\alpha}} X^{(n)}. \]
Then decomposition 2 can be rewritten as
\[ (3) \quad X^{(n)} = (\Sigma Y)^{(n)} \simeq \bigvee_\lambda \left( \bigvee_{j=1}^{d_\lambda} Q_\lambda(Y) \right), \]
where $\lambda$ runs over all $p$-regular partitions of $n$. Let $\bar{H}_*(X)$ denote the reduced mod $p$ homology.

**Theorem 1.1.** Let $X = \Sigma Y$ be a $p$-local suspension. Suppose that $p = 2$ or $\bar{H}_*(X)^{\text{odd}} = 0$. Then $Q_\lambda(Y)$ is contractible if and only if $\text{Len}(\lambda) > \dim \bar{H}_*(X)$.

**Note:** If $p > 2$ and $\bar{H}_*(X)^{\text{odd}} \neq 0$, then $Q_\lambda(Y)$ may not be contractible even if $\text{Len}(\lambda) > \dim \bar{H}_*(X)$. For instance, let $Y$ be an even sphere and let $X = \Sigma Y$, then $Q_{(1,1)}(Y) \simeq X^{(2)}$. 

From now let on, let \( p = 2 \). Let \( X = \Sigma Y \) be the suspension of a two-cell complex localized at 2. By Theorem 1.1, we have

\[
X^{(n)} \simeq \bigvee_{n/2 < \alpha \leq n} \left( \bigvee_{j=1}^{c_{(\alpha, n-\alpha)}} Q_{(\alpha, n-\alpha)}(Y) \right).
\]

We will determine the homology of \( Q_{(\alpha, n-\alpha)}(Y) \) for each \( n/2 < \alpha \leq n \) and the multiplicity \( d_{(\alpha, n-\alpha)} \).

Let \( u, v \) be a basis for \( V = \tilde{H}_*(X) \) with \( |u| \leq |v| \). Let \( S(V) \) be the free commutative algebra generated by \( V \) and let \( P(V) \) be the set of primitive elements in \( S(V) \). Then

\[
P(V) = \bigoplus_{k=0}^{\infty} P_{2^k}(V)
\]

and \( P_{2^k}(V) \) has a basis \( u^{2^k}, v^{2^k} \). Since \( V \) is a module over the Steenrod algebra, \( P_{2^k}(V) \) is a module over the Steenrod algebra. Let \( s(n) \) be the integer such that \( 2^{s(n)} - 1 \leq n \leq 2^{s(n)+1} - 2 \) and let

\[
n - 2^{s(n)} + 1 = 2^{a(1;n)} + 2^{a(2;n)} + \ldots + 2^{a(l(n);n)}
\]

with \( 0 \leq a(1;n) < a(2;n) < \ldots < a(l(n);n) \).

**Theorem 1.2.** Let \( X = \Sigma Y \) be a 2-local suspension such that \( V = \tilde{H}_*(X) \) has a basis \( |u|, |v| \). Then

1) there is an isomorphism of modules over the Steenrod algebra

\[
\tilde{H}_*(Q_{(n)}(Y)) \cong \bigotimes_{j=0}^{s(n)-1} P_{2^j}(V) \otimes \bigotimes_{j=1}^{l(n)} P_{2^{s(j;n)}}(V);
\]

2) there is a homotopy equivalence

\[
Q_{(\alpha, n-\alpha)}(Y) \simeq \sum_{[u]+[v]}^{(n-\alpha)} Q_{(n-2(n-\alpha))}(Y)
\]

for \( n/2 < \alpha \leq n \).

3) if \( \tilde{H}_*(X) \) has a non-trivial Steenrod operation (a Hopf invariant one complex) then the (mod 2) homology of the decomposition 4 is a complete decomposition over the Steenrod algebra.

This theorem follows from Proposition 4.4, Theorems 4.6, 5.7 and 5.8.

Assertion 3 says that any idempotent \( \phi: \tilde{H}_*(X^{(n)}) \to \tilde{H}_*(X^{(n)}) \) over the Steenrod algebra can be geometrically realized by an idempotent \( f: X^{(n)} \to X^{(n)} \). Furthermore the map \( f \) can be chosen to be functorial. Therefore

\[
f = \sum_{\sigma \in S_n} k_{\sigma} \sigma: X^{(n)} \to X^{(n)}
\]
for some $k_\sigma \in \mathbb{Z}_{(p)}$.
For each $a$ with $n/2 < a \leq n$, we have
\[ 2^{s(2a-n)} - 1 \leq 2a - n \leq 2^{s(2a-n)+1} - 2. \]
Let
\[ 2a - n - 2^{s(2a-n)} + 1 = \eta_0 + \eta_1 + \cdots + \eta_{s(2a-n) - 1} \cdot 2^{s(2a-n) - 1} \]
be the 2-adic decomposition, where $\eta_k = 0, 1$. Let
\[ \omega(a) = (\eta_0 + 1, \eta_1 + 1, \cdots, \eta_{s(2a-n) - 1} + 1). \]
For a non-negative integer $m$, define
\[
\epsilon(m) = \begin{cases} 
2 & \text{if } m > 0 \text{ and } m \text{ is even,} \\
1 & \text{if } m \text{ is odd,} \\
0 & \text{if } m = 0.
\end{cases}
\]
For a sequence of non-negative integers, $I = (i_1, \cdots, i_s)$, let
\[ \epsilon(I) = (\epsilon(i_1), \cdots, \epsilon(i_s)). \]

**Theorem 1.3.** For each $n/2 < a \leq n$, the multiplicity $d_{(a,n-a)}$ of $Q_{(a,n-a)}$ is given by the formula
\[
d_{(a,n-a)} = \sum_{I = (i_0 = n, \cdots, i_s)} k_I,
\]
where
\[
k_I = \left( \frac{i_0 - \epsilon(i_0)}{2} \right) \cdot \left( \frac{i_1 - \epsilon(i_1)}{2} \right) \cdots \left( \frac{i_s - \epsilon(i_s)}{2} \right) \cdot 2^{n - (\epsilon(i_0) + \epsilon(i_1) + \cdots + \epsilon(i_s))}. \]

This Theorem follows from Proposition 4.4, Theorems 5.7 and 5.8.

We should point out that Theorems 5.4 and 5.6 suggest that we can filter the symmetric group algebra and its decomposition matrix according to the length of partitions. Theorems 6.1 and 6.4 give an explicit formula to compute entries in the submatrix of the decomposition matrix of symmetric groups corresponding to two-row Young diagrams. This gives a different approach from the classical methods in the representation theory of symmetric groups.
The article is organized as follows. In Section 2, we establish a relation between functorial decompositions of tensor products and the representation of symmetric groups on the tensor products. We give some relations between the formal Steenrod algebras and the hyper algebras in Section 3. In Section 4, we give an explicit decomposition of self tensor products of two-dimensional modules and show that this decomposition is complete over the Steenrod algebra. We review and give some general results on representations of symmetric groups in Section 5. The proof of Theorem 1.1 is given in this section. In Section 6, we give some applications. In particular, we have Theorems 6.1 and 6.4 in this section.

2. Functorial Decompositions of Tensor Products

In this section, the ground ring \( \mathbf{k} \) is a field. Let \( V \) be a graded (ungraded) module and let \( S_n \) acts on \( V^{\otimes n} \) by permuting positions in graded (ungraded) sense. Let

\[
1 = \sum_{i=1}^{t} e_i
\]

be an orthogonal decomposition of the identity in terms of idempotents in \( \mathbf{k}(S_n) \). Let

\[
e_i(V) = \text{Im}(e_i: V^{\otimes n} \rightarrow V^{\otimes n}).
\]

Then there is a decomposition

\[
V^{\otimes n} \cong \bigoplus_{i=1}^{t} e_i(V).
\]

**Definition 2.1.** Let \( V \) be a fixed module. A decomposition

\[
V^{\otimes n} \cong \bigoplus_{i=1}^{t} W_i
\]

is called a **functorial** decomposition if there exist idempotents \( e_i \in \mathbf{k}(S_n) \) for \( 1 \leq i \leq t \) such that

1) there is an orthogonal decomposition of the identity

\[
1 = \sum_{i=1}^{t} e_i
\]

in \( \mathbf{k}(S_n) \);

2) there is an isomorphism of modules

\[
e_i(V) \cong W_i
\]

for each \( 1 \leq i \leq t \).
Let $T_n(V) = V^\otimes n$. According to the following easily checked theorem, for ungraded modules $V$, functorial decompositions of $V^\otimes n$ are equivalent to decompositions of the functor $T_n$.

**Theorem 2.2.** Let $V$ be a fixed ungraded module. Then

$$V^\otimes n \cong \bigoplus_{i=1}^t W_i$$

is a functorial decomposition if and only if there is a decomposition of the functor $T_n$, from ungraded modules to ungraded modules,

$$T_n \cong \bigoplus_{i=1}^t A_i$$

such that

$$A_i(V) \cong W_i$$

for $1 \leq i \leq t$.

There is a complete functorial decomposition

$$V^\otimes n \cong \bigoplus_{\lambda} Q_\lambda(V)^{\otimes d_\lambda}.$$ 

Let $X = X \Sigma Y$ be a suspension and let $V = \overline{H}_s(X; k)$. Then

$$\overline{H}_s(Q_\lambda(Y)) \cong Q_\lambda(V)$$

and so the homology of $Q_\lambda(Y)$ can be studied by examining functorial decompositions of $V^\otimes n$.

One natural tool for studying functorial decomposition of $V^\otimes n$ is the use of Hopf algebra actions on $V^\otimes n$, described as follows. Let $A$ be a graded Hopf algebra and let $V$ be a graded module over $A$. Then $V^\otimes n$ is a module over $A$ via the composite

$$A \otimes V^\otimes n \xrightarrow{\psi^n \otimes V^\otimes n} A^\otimes n \otimes V^\otimes n \cong (A \otimes V)^\otimes n \rightarrow V^\otimes n.$$

**Proposition 2.3.** Let $A$ be a graded cocommutative Hopf algebra and let $V$ be a graded module over $A$. Then any functorial decomposition of $V^\otimes n$ is a decomposition of $V^\otimes n$ over $A$.

The proof is straightforward.

In particular, if $V$ is a module over the Steenrod algebra (whatever geometrically realizable or not), then any functorial decomposition of $V^\otimes n$ is a decomposition over the Steenrod algebra. We will show that functorial decompositions of $V^\otimes n$ are the same as decompositions of $V^\otimes n$ over the Steenrod algebra when $p = 2$, $\dim V = 2$ and $V$ has a non-trivial Steenrod operation.
3. Hyperalgebras and formal Steenrod algebras

In this section, the ground ring $k$ is any field.

3.1. Cofree Bi-algebra Cover of an Algebra. Let $M$ be a graded module of finite type. Let $\Gamma(M) = S(M^*)^*$ be the (graded) dual of the free commutative (graded) algebra generated by the (graded) dual of $M$. Then $\Gamma(M)$ is a (co-free) coalgebra. Let $q_M : \Gamma(M) \to M$ be the dual of the inclusion $M^* \to S(M^*)$. By the universal property of the functor $S$, we have that $\Gamma(M)$ satisfies the following universal property:

Let $C$ be any cocommutative algebra and let $f : C \to M$ be any $k$-linear map. Then there exists a unique morphism of coalgebras $\tilde{f} : C \to \Gamma(M)$ such that $q_M \circ \tilde{f} = f$.

$\Gamma$ defines a functor from modules of finite type to cocommutative coalgebras of finite type. Now let $M$ be a general free graded module. Then $M$ is a colimit of submodules of finite type. That is, $M = \colim_\alpha M_\alpha$, where $M_\alpha$ runs through submodules of $M$ of finite type. We define

$$\Gamma(M) = \colim_\alpha \Gamma(M_\alpha).$$

The quotient map $q_{M_\alpha} : \Gamma(M_\alpha) \to M_\alpha$ induces the quotient map $q_M : \Gamma(M) \to M$. The functor $\Gamma$ is just the adjoint functor of the forgetful functor from cocommutative coalgebras to modules. The following proposition is well-known.

**Proposition 3.1.** The coalgebra $\Gamma(M)$ with the quotient map $q_M : \Gamma(M) \to M$ satisfies the following universal property:

Let $C$ be any cocommutative coalgebra and let $f : C \to M$ be any $k$-linear map. Then there exists a unique morphism of coalgebras $\tilde{f} : C \to \Gamma(M)$ such that $q_M \circ \tilde{f} = f$.

**Proof.** Since any cocommutative coalgebra is a colimit of subcoalgebras of finite type (See [12]), $C = \colim_\alpha C_\alpha$ with $C_\alpha$ subcoalgebras of $C$ of finite type. For each $C_\alpha$, $f(C_\alpha)$ is contained in a submodule of $M$ of finite type. Thus there is a unique morphism of coalgebras $\tilde{f}_\alpha : C_\alpha \to \Gamma(M)$ such that $q_M \circ \tilde{f}_\alpha = f|_{C_\alpha}$ for each $\alpha$. Now the morphism $\{\tilde{f}_\alpha\}$ defines a unique morphism of coalgebras $\tilde{f} : C \to \Gamma(M)$ and hence the result.

Let $x$ be homogeneous. Then the cofree cocommutative coalgebra $\Gamma(x)$ is given as follows.
1) If $k$ is of characteristic 2 or $|x|$ is even, then $\Gamma(x)$ has a basis $\{\gamma_j(x) | j \geq 0\}$ with $\gamma_0(x) = 1$, the co-unit, and

$$
\psi(\gamma_n(x)) = \sum_{i+j=n} \gamma_i(x) \otimes \gamma_j(x).
$$

2) If $k$ is of characteristic $\neq 2$ and $|x|$ is odd, then $\Gamma(x)$ has a basis $\{\gamma_j(x) | j = 0, 1\}$ with $\gamma_0(x) = 1$, the co-unit, and $\psi(\gamma_1(x)) = 1 \otimes \gamma_1(x) + \gamma_1(x) \otimes 1$.

Let $\{x_\alpha\}$ be a (homogeneous) basis for $M$. Then there is an isomorphism of coalgebras

$$
\Gamma(M) \cong \bigotimes_\alpha \Gamma(x_\alpha).
$$

Now suppose that $A$ is an algebra. Let $\mu: A \otimes A \to A$ be the multiplication and let $\eta: k \to A$ be the unit map. By the universal property of the functor $\Gamma$, we have a unique morphism of coalgebras $\mu: \Gamma(A) \otimes \Gamma(A) \to \Gamma(A)$ such that the following diagram commutes

$$
\begin{array}{ccc}
\Gamma(A) \otimes \Gamma(A) & \xrightarrow{\mu} & \Gamma(A) \\
\downarrow q_A \otimes q_A & & \downarrow q_A \\
A \otimes A & \xrightarrow{\mu} & A
\end{array}
$$

and a unique morphism of coalgebras $\eta: k \to \Gamma(A)$ such that $q_A \circ \eta = \eta: k \to A$. Thus $\Gamma(A)$ is a bialgebra which satisfies the following universal property.

**Proposition 3.2.** Let $A$ be an algebra. Then $\Gamma(A)$ satisfying:

Let $B$ be any cocommutative bialgebra and let $\phi: B \to A$ be any morphism of algebras. Then there is unique morphism of bialgebras $\tilde{\phi}: B \to \Gamma(A)$ such that $q_A \circ \tilde{\phi} = \phi$.

The proof from the universal property of the functor $\Gamma$.

### 3.2. The Graded Hyperalgebra and the Formal Steenrod Algebra

Let $V$ be a graded $k$-module. Let $\text{End}(V)$ be the set of graded endomorphisms of $V$. That is,

$$
\text{End}(V)_k = \{f: V \to V | f(V_i) \subseteq V_{i+k} \text{ for each } i\}.
$$

Then $\text{End}(V)$ becomes a graded (finite) algebra under composition and $V$ becomes a left $\text{End}(V)$-module. Since $q: \Gamma(\text{End}(V)) \to \text{End}(V)$ is a morphism of algebras, $V$ is a module over $\Gamma(\text{End}(V))$. It follows that $V^{\otimes n}$ is module over $\Gamma(\text{End}(V))$ via
action described earlier utilizing the comultiplication. We write $\mathcal{U}^V$ for $\Gamma(\text{End}(V))$. By Proposition 3.2, the bialgebra $\mathcal{U}^V$ satisfies the following universal property.

Let $B$ be a cocommutative graded bialgebra. If $V$ is a module over $B$, then there is a unique morphism of bi-algebras $\phi: B \to \mathcal{U}^V$ such that

$$\mu_B = \mu_{\mathcal{A}^V} \circ (\phi \otimes V): B \otimes V \to V.$$  

Equation 7 gives the basic formula for calculations of $\mathcal{U}^V$-action on $V^\otimes n$. One can easily show that if $V$ is an ungraded $k$-module, then $\mathcal{U}^V$ is the Hyperalgebra defined in [1, 11]. For this reason, we call $\mathcal{U}^V$ the Hyperalgebra induced by $V$ whether $V$ is a graded or ungraded $k$-module.

Let $\text{End}(V)^+ = \bigoplus_{k \geq 0} \text{End}(V)_k$ and let $\text{End}(V)^-_k = \text{End}(V)^+_{-k}$. Then

$$\text{End}(V) = \text{End}(V)^+ \oplus \text{End}(V)_0 \oplus \text{End}(V)^-$$

as graded modules and so there is an isomorphism of coalgebras

$$(8) \quad \mathcal{U}^V = \Gamma(\text{End}(V)) \cong \Gamma(\text{End}(V)^+) \otimes \Gamma(\text{End}(V)_0) \otimes \Gamma(\text{End}(V)^-) .$$

Since each of $\text{End}(V)^+$, $\text{End}(V)_0$ and $\text{End}(V)^-$ is closed under the multiplication of $\text{End}(V)$, each of $\Gamma(\text{End}(V)^+)$, $\Gamma(\text{End}(V)_0)$ and $\Gamma(\text{End}(V)^-)$ is a sub bialgebra of $\mathcal{U}^V$. Let $\mathcal{A}^V$ denote $\Gamma(\text{End}(V)^+)$. Then $\mathcal{A}^V$ is a connected graded bialgebra and so it is a connected Hopf algebra. We call $\mathcal{A}^V$ the formal Steenrod algebra induced by $V$. By Formula 8, the action of $\mathcal{U}^V$ on $V^\otimes n$ is generated by

1) the action of $\mathcal{A}^V$;
2) the action of $\Gamma(\text{End}(V)_0)$;
3) the dual action of $\mathcal{A}^V$.

Let $V$ be a graded module such that $\dim V_k \leq 1$ for each $k$ and let $\{x_i\}_{i \in I}$ be a homogeneous basis for $V$, where $I$ is an ordered subset of $\mathbb{Z}$. For each pairing $(i, j)$ with $i, j \in I$, let $e_{ij} \in \text{End}(V)$ be the elementary matrix such that

$$e_{ij}(x_k) = \delta_{jk}x_i$$

for $k \in I$, where $\delta_{st} = 1$ if $s = t$ and $\delta_{st} = 0$ if $s \neq t$. The degree of $e_{ij}$ is given by

$$|e_{ij}| = |x_i| - |x_j| .$$

Let $w = x_{i_1} \cdots x_{i_n}$ be a monomial in $V^\otimes n$ and let

$$c_i(w) = |\{i_k = i|1 \leq k \leq n\}|$$

the number of occurrences of $x_i$ in $w$. Then

$$(9) \quad \gamma_k(e_{ii})(w) = \binom{c_i}{k} w$$
for each \( k \geq 0 \). Let \( \mathbf{k} \) be of characteristic \( p \) with \( p \) finite or infinite and let \( c_I = \{c_i\}_{i \in I} \) be an ordered set of non-negative integers and let \( \mathcal{V}(c_I; n) \) be the submodule of \( \mathcal{V}^{\otimes n} \) spanned by the monomial \( w \) with \( c_i(w) = c_i \) for each \( i \in I \). By Formula 9, we have

**Proposition 3.3.** Let \( \mathcal{V} \) be a graded module. Suppose that \( \mathcal{V} \) has at most one cell in each dimension. Then \( \phi: \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes n} \) is a morphism over \( \Gamma(\text{End}(\mathcal{V})) \) if and only if \( \phi \) maps \( \mathcal{V}(c_I; n) \) into \( \mathcal{V}(c_I; n) \) for each ordered set of non-negative integers \( c_I \).

A \( \mathbf{k} \)-linear endomorphism of \( \mathcal{V}^{\otimes n} \) which sends each \( \mathcal{V}(c_I; n) \) to itself is called a weighted map in [9]. Note that \( \mathcal{V}^{\otimes n} \) is a weighted bi-module over \( \mathcal{A}^V \). That is, \( \mathcal{V}^{\otimes n} \) is a left and right \( \mathcal{A}^V \)-module, where the right \( \mathcal{A}^V \) action on \( \mathcal{V}^{\otimes n} \) is given by the dual action of \( \mathcal{A}^V \). By Formula 8, we have the following connection between the Hyperalgebra and the formal Steenrod algebra.

**Proposition 3.4.** Let \( \mathcal{V} \) be graded module with a graded basis \( \{x_i | i \in I\} \). Suppose that \( \mathcal{V} \) has at most one cell in each dimension. Let \( \phi: \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes n} \). Then \( \phi \) is a morphism of modules over \( \mathcal{U}^V \) if and only if \( \phi \) is a morphism of weighted bi-modules over \( \mathcal{A}^V \).

**Remark 3.5.**
1) If \( \text{dim}(\mathcal{V}) = 2 \) having a homogeneous basis \( u, v \) with \( |u| < |v| \), then a map \( \phi: \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes n} \) is a weighted map if and only if \( \phi \) is a graded map. In this case, \( \phi: \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes n} \) is a morphism over \( \mathcal{U}^V \) if and only if \( \phi \) is a morphism of graded bi-modules over \( \mathcal{A}^V \);
2) Let \( \mathcal{V} \) be an ungraded module with \( \text{dim}(\mathcal{V}) = k \). Let \( \{x_1, \ldots, x_k\} \) be a basis for \( \mathcal{V} \). Given an positive integer \( n \), we can choose a grading on \( \mathcal{V} \) such that the following property holds:
   A map \( \phi: \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes n} \) is a weighted map if and only if \( \phi \) is a graded map.
   For instance, we can choose a grading on \( \mathcal{V} \) such that \( |x_1| < |x_2| \) and \( n|x_j| < |x_{j+1}| \) for \( 2 \leq j \leq k - 1 \).

Let \( \mathcal{V} \) be an ungraded module with \( \text{dim}(\mathcal{V}) = k \). Then \( \mathcal{V} \) is a module over the general linear group \( GL(k; \mathbf{k}) \) and so is \( \mathcal{V}^{\otimes n} \) for any \( n \). Let \( |\mathbf{k}| \) denote the cardinality of the set \( \mathbf{k} \).

**Theorem 3.6.** [11, Theorems 8.2.13 and 8.2.14] Let \( \mathcal{V} \) be an ungraded module with \( \text{dim}(\mathcal{V}) = k \). Then
1) Every \( \mathcal{U}^V \)-submodule of \( \mathcal{V}^{\otimes n} \) is a \( GL(k; \mathbf{k}) \)-submodule.
2) If \( |\mathbf{k}| \geq n + 2 \), every \( GL(k; \mathbf{k}) \)-submodule of \( \mathcal{V}^{\otimes n} \) is a \( \mathcal{U}^V \)-submodule.
3) If \( W \) is a finitely generated \( \mathcal{U}^V \)-submodule of \( \mathcal{V}^{\otimes n} \), then every \( \mathcal{U}^V \)-homomorphism from \( W \) into \( T(V) \) is a \( GL(k; \mathbf{k}) \)-homomorphism;
4) If \( |\mathbf{k}| \geq \max\{m, n + 2\} \) and \( W \) is a \( GL(k; \mathbf{k}) \)-submodule of \( \mathcal{V}^{\otimes n} \), then every \( GL(k; \mathbf{k}) \)-homomorphism from \( W \) into \( \mathcal{V}^{\otimes m} \) is a \( \mathcal{U}^V \)-homomorphism.
4. Decompositions of \( V \otimes n \) over Hopf algebras

In this section, the ground field \( k \) is a field of characteristic 2 and \( V \) is a 2-dimensional module over the formal Steenrod algebra \( \mathcal{A} \). Let \( \{u, v\} \) be a basis for \( V \) with \(|u| = 0\) and \(|v| = 1\). Let \( \{e_{12}\} \) and \( \{e_{21}\} \) be the basis for \( \text{End}(V)^+ \) and \( \text{End}(V)^- \), respectively. We write \( S_q \) for the element \( \gamma_k(e_{12}) \in \mathcal{U}_k^V \) and \( S_q^* \) for \( \gamma_k(e_{21}) \in \mathcal{U}_k^V \). In particular, \( S_q(u) = v \) and \( S_q^*(v) = u \). Let \( W \) be any module. Let \( L_n(W) \) denote the set of Lie elements of degree \( n \) in the free Lie algebra \( L(W) \). Let \( a_1, \ldots, a_n \) be elements in \( W \). We write \( \beta_n(a_1a_2\cdots a_n) = [[a_1, a_2, \ldots, a_n]] \) for the \( n \)-fold iterated commutator from left to right. Let \( P_2^k(W) \) be the set of primitive elements of degree \( 2^k \) in the symmetric algebra \( S(W) \). Note \( L_1(V) = P_1(V) = V \).

**Proposition 4.1.** Let \( V \) be a 2-dimensional module. Then there are isomorphisms of modules over \( \mathcal{U}^V \):

1) \( L_2(P_2^k(V)) \cong P_2^k(L_2(V)) \cong L_2(V)^{\otimes k+1} \) is a one-dimensional module for any \( k \geq 0 \);
2) \( P_2^k(P_2^l(V)) \cong P_2^{k+l}(V) \) is a two-dimensional module for any \( k, l \geq 0 \);
3) \( L_3(V) \cong L_2(V) \otimes V \);
4) \( S_3(V) \cong P_2(V) \otimes V \).

The proof is immediate.

**Remark 4.2.** Of course \( P_2(V) \) and \( V \) are isomorphic as modules over \( \mathbb{Z}/2[S_2] \), but the action of the Steenrod algebra is different. \( P_2(V) \) corresponds to the Frobenious kernel \( V^F \), and part (4) above corresponds to Donkin’s factorization which in his notation was written as \( M(3) \cong M(1)^F \otimes M(1) \).

**Lemma 4.3.** Let \( V \) be a 2-dimensional module. Then there is a complete decomposition over \( \mathcal{U}^V \)

\[
P_{2^k}(V)^{\otimes 3} \cong ((L_2(V)^{\otimes k+1})^{\otimes 2} \oplus P_{2k+1}(V)) \otimes P_{2^k}(V)
\]

for any \( k \geq 0 \).

The proof is immediate.

Let \( m \) be a non-negative integer. Let \( \epsilon(m) \) be defined by equation 5. Let \( B \) and \( C \) be functors. We write \( B^\otimes \) for \( B^\otimes \) (including \( B^0 = k \)) and \( BC \) for \( B \otimes C \). We write \( kB \) for \( B^{\otimes k} \) (including \( 0B = 0 \)). In the following proposition, \( P_{2^*} = P_{2^*}(V) \), \( L_i = L_i(V) \) and \( L_i \circ P_{2^*} = L_i(P_{2^*}(V)) \).
Proposition 4.4. There is a decomposition over $U^V$
\[ V^{\otimes n} \cong \bigoplus k_i B_I \cdot P_2^{i_2} \cdot P_2^{i_2+i_3} \cdots P_2^{i_{s+1}} P_1^{(n)} , \]
where
\[ B_I = \frac{n-i(n)}{2} \cdot (L_2 \circ P_2)^{-\frac{i_2-i(i_2)}{2}} \cdots (L_2 \circ P_2)^{-\frac{i_{s+1}-i(i_{s+1})}{2}} , \]
and
\[ k_I = \left( \frac{n-i(n)}{2} \right) \cdot \left( \frac{i_2-i(i_2)}{2} \right) \cdots \left( \frac{i_{s+1}-i(i_{s+1})}{2} \right) . \]

Note: $\dim B_I = 1$ and so $B_I \otimes M$ is just a suspension of $M$.

Proof. Notice that $V^{\otimes 3} \cong (2L_2 \oplus P_2) \otimes P_1$. It follows that
\[ V^{\otimes 2k+1} \cong (2L_2 \oplus P_2)^k \otimes P_1 \quad \text{and} \quad V^{\otimes 2k+2} \cong (2L_2 \oplus P_2)^k \otimes P_1^2 . \]
Thus
\[ V^n \cong \bigoplus_{i=0}^{n-i(n)} 2^{n-i(n)-i} \left( \frac{n-i(n)}{2} \right) \cdot \frac{n-i(n)}{2} \frac{n-i(n)}{2} \cdots P_2^{i_2} P_1^{(n)} . \]

Notice that $\dim P_2 = 2$ for any $l \geq 0$ with a basis $u^{2l}$ and $v^{2l}$ and the formal Steenrod operation given by $S_q^r (v^{2l}) = u^{2l}$. It follows that
\[ P_2^{k} \cong \bigoplus_{i=0}^{k-i(k)} 2^{k-i(k)-i} \left( \frac{k-i(k)}{2} \right) \cdot \frac{k-i(k)}{2} \frac{k-i(k)}{2} \cdots P_2^{i_2} P_2^{i_3} P_1^{(k)} , \]
for any $l, k$. Observe that $P_2 \circ P_2 \cong P_2^{k+1}$. The assertion follows. \[ \square \]

Now we are going to show that this is a complete decomposition over the formal Steenrod algebra $A^V$. It suffices to show that each $P_2^{i_1} (V) \otimes \cdots \otimes P_2^{i_n} (V) \otimes V^{(n)}$ is indecomposable over $A^V$, where $i(i_j) = 1$ or 2.

Lemma 4.5. Let $0 \leq c_1 < c_2 < \cdots < c_k$ and let $\epsilon_j = 1$ or 2 for $1 \leq j \leq k$. Then
\[ M = P_2^{c_1} (V)^{\otimes \epsilon_1} \otimes P_2^{c_2} (V)^{\otimes \epsilon_2} \otimes \cdots \otimes P_2^{c_k} (V)^{\otimes \epsilon_k} \]
is an indecomposable module over the Steenrod algebra.
Proof. The proof is by induction on $k$. The assertion holds when $k = 1$. Suppose that the assertion holds for all sequences $0 < c_1 < \cdots < c_k'$, with $k' < k$ and $k \geq 2$. Let $0 < c_1 < \cdots < c_k$. Let $f : M \to M$ be any idempotent over the Steenrod algebra such that $f$ sends the top cell $v^{2^e_1+\cdots+2^e_k+e_k}$ to itself. We show that $f$ is an isomorphism from which the assertion will follow. Let $N = P_{2^{e_2}} (V)^{\otimes 2} \otimes \cdots \otimes P_{2^{e_k}} (V)^{\otimes e_k}$. Let

$$C = \text{Im}(Sq_*^{2^{e_1}} : P_{2^{e_1}} (V) \to P_{2^{e_1}} (V)) = \text{Ker}(Sq_*^{2^{e_1}} : P_{2^{e_1}} (V) \to P_{2^{e_1}} (V)).$$

Notice that $Sq_*^{2^{e_1}} : N \to N$ is zero. Thus

$$\text{Im}(Sq_*^{2^{e_1}} : M \to M) = \text{Ker}(Sq_*^{2^{e_1}} : M \to M) = C \otimes N.$$

Notice that $f : M \to M$ is morphism over the Steenrod algebra. Thus $f$ sends $M_1 \otimes N$ into itself. Let $f' : C \otimes N \to C \otimes N$ be the resulting map. Notice that $C$ is a submodule of $P_{2^{e_1}} (V)$ over the Steenrod algebra. Thus $C \otimes N$ is a submodule of $M$ over the Steenrod algebra and $f' : C \otimes N \to M_2 \otimes N$ is a morphism over the Steenrod algebra. We consider two cases:

**Case I.** $\epsilon_1 = 1$.

Then $C$ is a one-dimensional $k$-module generated by $u^{2^e_1}$. Notice that

$$Sq_*^{2^e_1+\cdots+2^e_k+e_k} (v^{2^e_1+\cdots+2^e_k+e_k}) = u^{2^e_1+\cdots+2^e_k+e_k},$$

by the Cartan formula. Thus $f' : C \otimes N \to C \otimes N$ sends the bottom cell $u^{2^e_1+\cdots+2^e_k+e_k}$ to itself. Notice that $C \otimes N \cong N$ as modules over the Steenrod algebra. By induction, $f'$ is an isomorphism. Let $D = P_{2^{e_1}} (V) / C$. Then there is a commutative diagram of short exact sequences over the Steenrod algebra

$$\begin{array}{ccc}
C \otimes N & \to & M \\
\uparrow f' & & \downarrow f \\
C \otimes N & \to & M \\
\downarrow f'' & & \downarrow f'' \\
C \otimes N & \to & M \\
\end{array}$$

Notice that $f'' : D \otimes N \to D \otimes N$ sends the top cell $u^{2^e_1+\cdots+2^e_k+e_k}$ to itself. Thus $f'' : D \otimes N \to D \otimes N$ is an isomorphism by induction and so $f$ is an isomorphism by the five-lemma.

**Case II.** $\epsilon_1 = 2$.

Then $C$ is generated by $u^{2^e_1+1}$ and $[u^{2^e_1}, v^{2^e_1}] = u^{2^e_1} v^{2^e_1} - v^{2^e_1} u^{2^e_1}$. Let $C^{(1)}$ and $C^{(2)}$ be the submodules of $C$ generated by $u^{2^e_1+1}$ and $u^{2^e_1} v^{2^e_1} + v^{2^e_1} u^{2^e_1}$, respectively. Then $C \otimes N = C^{(1)} \otimes N \oplus C^{(2)} \otimes N$ as modules over the Steenrod algebra. Let $n$ be an integer such that $C^{(1)} \otimes N$ is not zero in dimension $n$. Then

$$n = 2(2^e_1 + \cdots + 2^e_k) + 2^e_1 + \cdots + 2^e_k.$$
for some sequence $2 \leq i_1 < i_2 < \cdots < i_t \leq c_k$, $2 \leq j_1 < i_2 < \cdots < j_s \leq c_k$ with $\{i_1, \ldots, i_t\} \cap \{j_1, \ldots, j_s\} = \emptyset$ determined by the occurrences of $v$’s. In particular, $n$ is divisible by $2^{c_1+1}$. Let $m$ be integer such that $C^{(2)} \otimes N$ is not zero in dimension $m$. Then $m = 2(2^{c_1} + m')$, where $m'$ is divisible by $2^{c_2}$. In particular, $m$ is not divisible by $2^{c_2+1}$ because $c_1 < c_2$. It follows that $f$ sends $C^{(1)} \otimes N$ and $C^{(2)} \otimes N$ into themselves for dimensional reasons. Let $f^{(i)} : C^{(i)} \otimes N \to C^{(i)} \otimes N$ be the resulting map for $i = 1, 2$. We need to show that $f^{(1)}$ and $f^{(2)}$ are nontrivial. Notice that $f$ sends the bottom cell to the bottom cell. Thus $f^{(1)}$ sends the bottom cell to the bottom cell and so $f^{(1)} : C^{(1)} \otimes N \cong N \to C^{(1)} \otimes N \cong N$ is nontrivial and thus an isomorphism. Notice that

$$S_{q_{a}}^{2^{c_1}+m}(v^{2^{c_1}+1+m}) = (u^{2^{c_1}} v^{2^{c_1}} + v^{2^{c_1}} u^{2^{c_1}})u^{m},$$

where $m = 2^{c_1+c_2} + \cdots + 2^{c_k+c_k}$. Thus $f^{(2)} : C^{(2)} \otimes N \to C^{(2)} \otimes N$ is nontrivial and thus an isomorphism. Let $E = P_{2^{c_1}}(V)^{\otimes 2}/C$. Then there is a short exact sequence of modules over the Steenrod algebra $C \otimes N \to M \to E \otimes N$. Let $f'' : E \otimes N \to E \otimes N$ be the morphism over the Steenrod algebra induced by $f : M \to M$. Now $E$ has a basis represented by $v^{2^{c_1+1}}$ and $u^{2^{c_1}} v^{2^{c_1}}$. Let $E^{(1)}$ and $E^{(2)}$ be the submodules of $E$ generated by $v^{2^{c_1+1}}$ and $u^{2^{c_1}} v^{2^{c_1}}$ respectively. Then $E \otimes N = E^{(1)} \otimes N \oplus E^{(2)} \otimes N$ as modules over the Steenrod algebra. By arguments similar to the preceding, $E^{(1)} \otimes N$ and $E^{(2)} \otimes N$ do not have nontrivial elements in the same dimension and so $f'' : E \otimes N \to E \otimes N$ can be decomposed into morphisms $\tilde{f}^{(1)} : E^{(1)} \otimes N \to E^{(1)} \otimes N$ and $\tilde{f}^{(2)} : E^{(2)} \otimes N \to E^{(2)} \otimes N$. Notice that $f$ sends the top cell to the top cell. Thus $\tilde{f}^{(1)}$ sends the top cell to the top cell and so $\tilde{f}^{(1)} : E^{(1)} \otimes N \to E^{(1)} \otimes N$ is an isomorphism. Now we need show that $\tilde{f}^{(2)} : E^{(2)} \otimes N \to E^{(2)} \otimes N$ is nontrivial. Notice that a basis for $M_t$ with $t \leq 2^{c_1}$ is given by:

$$u^{2^{c_1}+1}u^{m}, v^{2^{c_1}} u^{2^{c_1}} u^{m}, u^{2^{c_1}} v^{2^{c_1}} u^{m},$$

where

$$f(u^{2^{c_1}+1} u^{m}) = u^{2^{c_1}+1} u^{m} f(u^{2^{c_1}} v^{2^{c_1}} + v^{2^{c_1}} u^{2^{c_1}} u^{m}) = (u^{2^{c_1}} v^{2^{c_1}} + v^{2^{c_1}} u^{2^{c_1}})u^{m}$$

$$S^{q_{a}}_{v^{2^{c_1}} u^{2^{c_1}} u^{m}} = S^{q_{a}}_{v^{2^{c_1}} v^{2^{c_1}} u^{m}} = u^{2^{c_1}+1} u^{m}. $$

Let

$$f(v^{2^{c_1}} u^{2^{c_1}} u^{m}) = k_{11} v^{2^{c_1}} u^{2^{c_1}} u^{m} + k_{12} v^{2^{c_1}} v^{2^{c_1}} u^{m},$$

$$f(u^{2^{c_1}} v^{2^{c_1}} u^{m}) = k_{21} u^{2^{c_1}} v^{2^{c_1}} u^{m} + k_{22} u^{2^{c_1}} v^{2^{c_1}} u^{m}. $$

Then we have

$$k_{11} + k_{12} = 1, \quad k_{21} + k_{22} = 1, \quad k_{21} + k_{11} = 1, \quad k_{22} + k_{12} = 1. $$

Thus the determinant of $f : M_{2^{c_1}} \to M_{2^{c_1}}$ is $2k_{11} - 1$ and so $f : M_{2^{c_1}} \to M_{2^{c_1}}$ is an isomorphism. It follows that $\tilde{f}^{(2)} : E^{(2)} \otimes N \to E^{(2)} \otimes N$ is nontrivial and thus an
isomorphism. Now \( f'': E \otimes N \to E \otimes N \) is an isomorphism and so \( f: M \to M \) is an isomorphism. The induction is finished and the assertion follows. \( \square \)

Thus we have

**Theorem 4.6.** Let \( V \) be a 2-dimensional module. Then the decomposition given in Proposition 4.4 is a complete decomposition over \( \mathcal{A}^V \). In particular, it is a complete decomposition over \( \mathcal{U}^V \).

5. Relationship with Representations of \( S_n \)

For a vector space \( V \), write \( |V| \) for the dimension of \( V \). For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \), write \( \text{Len}(\lambda) = r \). We write \( S_\lambda \) for the subgroup \( S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_r} \) of \( S_n \) and let \( \xi_\lambda \) denote the representation of \( n \) induced from the trivial representation on \( S_\lambda \). We write \( \alpha_\lambda \) for the characteristic 0 irreducible corresponding to \( \lambda \). Thus \( \alpha_\lambda = \text{Im} \epsilon_\lambda \).

In this section it suffices to consider the case where \( k = \mathbb{Z}/p\mathbb{Z} \). The projective indecomposable \( k(S_n) \) modules are in \( 1 - 1 \) correspondence with the \( p \)-regular partitions of \( n \). Given a \( p \)-regular partition \( \lambda \), let \( Q_\lambda \) denote the corresponding projective and let \( f_\lambda \) denote the idempotent such that \( Q_\lambda = k(\Sigma_n)f_\lambda \). For a vector space \( V \), write \( Q_\lambda(V) = f_\lambda(V^\otimes n) \).

**Remark 5.1.** \( f_\lambda \) is not the mod \( p \) reduction of \( e_\lambda \).

For any projective \( Q \) in \( \mathbb{Z}/p\mathbb{Z}(G) \) there is a unique lift to a projective in \( \hat{Q} \) over \( \mathbb{Z}(p) \). We write \( \hat{Q} \) for \( \hat{Q} \otimes \hat{Q} \).

For partitions \( \lambda, \mu \) of \( n \), as in [11, p. 23] we define a partial order by \( \lambda \leq \mu \) if and only if \( \sum_{j=1}^t \lambda_j \leq \sum_{j=1}^t \mu_j \) for all \( t = 1, \ldots, \text{Len}(\lambda) \).

**Lemma 5.2.** If \( \lambda, \mu \) are partitions of \( n \) with \( \lambda \leq \mu \) then \( \text{Len}(\lambda) \geq \text{Len}(\mu) \).

**Proof.** If \( \text{Len}(\lambda) < \text{Len}(\mu) \) then for \( t = \text{Len}(\lambda) \) we have

\[
\sum_{j=1}^t \lambda_j = n = \sum_{j=1}^{\text{Len}(\mu)} \lambda_j > \sum_{j=1}^t \lambda_j,
\]

contradicting \( \lambda \leq \mu \). \( \square \)

For a partition \( \mu \) we define \( \mu^R \) as in [11, p. 282] and observe that \( \mu \leq \mu^R \) for any \( \mu \). According to [11, Theorem 6.3.51, p. 283], if the intertwining number \( (\hat{Q}_\lambda, \alpha_\mu) \) is nonzero then \( \mu^R \leq \lambda \) and so \( \mu \leq \lambda \) and therefore \( \text{Len}(\lambda) \leq \text{Len}(\mu) \).

**Theorem 5.3.** If \( \text{Len}(\lambda) > |V| \) then \( Q_\lambda(V) = 0 \).
Proof. Suppose \( \text{Len}(\lambda) > |V| \). Then \((\tilde{Q}_\lambda, \alpha_\mu) = 0\) for any partition \( \mu \) with \( \text{Len}(\mu) \leq |V| \). According to [11, Lemma 2.1.10, p. 37], the nonzero components of \( \xi_\tau \) in the basis of \( \alpha_\mu \)’s all satisfy \( \tau \leq \mu \). If \((\tilde{Q}_\lambda, \xi_\tau) \neq 0\), then \( \tau \) has a nonzero component \( \alpha_\mu \) such that \((\tilde{Q}_\lambda, \alpha_\mu) \neq 0\). For this \( \mu \) we have \( \tau \leq \mu \leq \lambda \), which implies that \( \text{Len}(\lambda) \leq \text{Len}(\tau) \). Therefore \((\tilde{Q}_\lambda, \xi_\tau) = 0\) for any partition \( \tau \) with \( \text{Len}(\tau) \leq |V| \). But according to [13, Theorem 8.1, p. 179] \((\tilde{Q}_\lambda, \xi_\tau) = 0\) equals the dimension of the space of elements of type \( \tau \) in \( \tilde{Q}_\lambda(V) \). Therefore \((\tilde{Q}_\lambda, \xi_\tau) = 0\) for all partitions \( \tau \) with \( \text{Len}(\tau) \leq |V| \) implies that \( Q_\lambda(V) = 0 \).

**Theorem 5.4.** Let \( k(\Sigma_n) = \sum d_\lambda Q_\lambda \) be the decomposition of the group ring into indecomposable projectives. Then \( V^{\otimes n} = \sum_{\text{Len}(\lambda) \leq |V|} d_\lambda Q_\lambda(V) \) is the complete natural decomposition of \( V^{\otimes n} \) into indecomposable pieces.

Proof. \( k(\Sigma_n) = \sum d_\lambda Q_\lambda \) implies that for any vector space \( V \), we have a natural decomposition \( V^{\otimes n} = \sum d_\lambda Q_\lambda(V) \) and so the preceding theorem implies that there is a natural decomposition \( V^{\otimes n} = \sum_{\text{Len}(\lambda) \leq |V|} d_\lambda Q_\lambda(V) \).

For any \( \lambda \), \( f_\lambda \) is primitive (not a nontrivial sum of orthogonal idempotents) so by the definition of a functorial decomposition (section 6.1) there is no natural decomposition of \( Q_\lambda(V) \). Therefore \( V^{\otimes n} = \sum_{\text{Len}(\lambda) \leq |V|} d_\lambda Q_\lambda(V) \) is the complete functorial decomposition of \( V^{\otimes n} \).

**Proof of Theorem 1.1.** By Theorems 5.3 and 5.4, we have \( \hat{H}_*(Q_\lambda(Y)) = 0 \) if and only if \( \text{Len}(\lambda) > \text{dim} \hat{H}_*(X) \). Since \( \hat{H}_*(X; \mathbb{Z}_{(p)}) \) is a finitely generated module over \( \mathbb{Z}_{(p)} \), \( \hat{H}_*(Q_\lambda(Y); \mathbb{Z}_{(p)}) \) is a finitely generated module over \( \mathbb{Z}_{(p)} \). It follows that \( \hat{H}_*(Q_\lambda(Y)) = 0 \) if and only if \( \hat{H}_*(Q_\lambda(Y); \mathbb{Z}_{(p)}) = 0 \), if and only if \( Q_\lambda(Y) \) is contractible. The assertion follows.

**Lemma 5.5.** Let \( V \) be a graded module with a basis \( \{ x_1, \cdots, x_k \} \). Let \( \lambda = (\lambda_1, \cdots, \lambda_s) \) be a \( p \)-regular partition with \( s \leq k \) and let \( t_\lambda = \lambda_1 |x_k| + \lambda_2 |x_{k-1}| + \cdots + \lambda_s |x_{k-s+1}| \).

Suppose that

1) \( |x_1| < |x_2| < \cdots < |x_k| \) and 
2) \( p = 2 \) or \( \text{V}^{\text{odd}} = 0 \).

Then \( Q_\lambda(V)_j = 0 \) for \( j > t_\lambda \) and \( \text{dim} Q_\lambda(V)_{t_\lambda} = 1 \).

Proof. Let \( \tilde{V} \) be an module over \( \mathbb{Q} \) with \( \text{dim} \tilde{V} = k \). According to [11, Chapter 8], the functorial indecomposable factors are classified by the Weyl modules \( W_\mu(\tilde{V}) \). By
the construction of the Weyl modules \([11, \text{Section 8.2}]\), the top degree of \(W_\mu(\tilde{V})\) is \(t_\mu\) for any partition \(\mu = (\mu_1, \ldots, \mu_t)\) with \(t \leq k\) and \(\dim W_\mu(\tilde{V}) = 1\).

As above \([13, \text{Theorem 8.1}]\), \(\tilde{Q}_\lambda(V)\) has elements of type \(\tau\) if and only if \((\tilde{Q}_\lambda, \xi_\tau) \neq 0\). Since \((\tilde{Q}_\lambda, \xi_\lambda) = 1\), \(\tilde{Q}_\lambda(V)\) has elements of type \(\lambda\) and the space of elements of type \(\lambda\) occurs once as a retract. If \(\tilde{Q}_\lambda(V)\) has elements of type \(\lambda'\) with \(\lambda' \neq \lambda\) then as above \(\lambda' \not\leq \lambda\) and therefore \(t_{\lambda'} < t_\lambda\). The assertion follows.

**Theorem 5.6.** Let \(V\) be an ungraded module with \(|V| = k\). Then
\[
\tilde{Q}_\lambda(V)
\]
is an indecomposable module over \(U^V\) if \(\text{Len}(\lambda) \leq k\).

**Proof.** Let \(k\) be an extension field of \(k\) such that \(|k|\) is infinite. According to \([9, \text{Theorem (3.5a)}]\) and \([11, \text{Theorem 8.3.8}]\), any irreducible factor of \(V^\otimes n\) is absolutely irreducible. It follows that \(Q_\lambda(V)\) is indecomposable over \(U^V\) if and only if
\[
Q_\lambda(V) \otimes_k k
\]
is indecomposable over \(U^V \otimes_k k\). Hence we may assume that the ground field itself is infinite. By Theorem 3.6, \(Q_\lambda(V)\) is indecomposable over \(U^V\) if and only if \(Q_\lambda(V)\) is indecomposable over \(GL(k, k)\). If \(k \geq n\), the assertion follows from \([10, \text{Theorem (2.2)}]\). Now we consider the degenerate case that \(k < n\). Let \(U\) be an \(n\)-dimensional module such that \(V\) is a submodule of \(U\). Then \(Q_\lambda(U)^*\) is a summand of \((U^*)^\otimes n \cong U^\otimes n\) as modules over \(GL(n, k)\). Then the socle \(\text{Soc}(Q_\lambda(U)^*)\) is a direct sum of irreducible submodules of \(U^\otimes n\) as modules over \(GL(n, k)\). According to \([11, \text{Theorem (2.8)}]\), \(\text{Soc}(Q_\lambda(U)^*)\) is an irreducible module because the socle of a primitive projective \(S_n\)-module is simple right ideal of \(k(S_n)\). Thus the head \(\text{Hd}(Q_\lambda(U))\) is an irreducible module over \(GL(n, k)\) and so \(Q_\lambda(U)\) has a unique maximal proper submodule as modules over \(GL(n, k)\). According to \([9, \text{pp.87}]\), the restriction
\[
Q_\lambda(V) = Q_\lambda(U) \cap V^\otimes n
\]
has a unique maximal proper submodule as modules over \(GL(k, k)\). It follows that \(Q_\lambda(V)\) is an indecomposable module over \(GL(k, k)\) and hence the result.

**Theorem 5.7.** Let \(k\) be of characteristic 2 and let \((a, n - a)\) be a 2-regular partition of \(n\). Then \(Q_{(a, n - a)}(V)\) is a functorial retract of
\[
Q_{2a - n}(V) \otimes L_2(V)^{\otimes n - a}.
\]

**Proof.** Let \(T_{ij}: V^\otimes n \rightarrow V^\otimes n\) be the \(k\)-linear map that switches positions \(i\) and \(j\). Let \(\phi: V^\otimes n \rightarrow V^\otimes n\) be the composite
\[
\phi = T_{2a - n, n - 1} \circ T_{2a - n, n - 3} \circ \cdots \circ T_{2a - n, 2a - n + 1}.
\]
Let $\theta : V \otimes n \to V \otimes n$ be the composite

$$V \otimes n \overset{\phi}{\longrightarrow} V \otimes n \overset{Q_{(2n-1)} \otimes 2 \otimes \cdots \otimes \alpha}{\longrightarrow} V \otimes n.$$ 

Let $A(V)$ be the colimit

$$A(V) = \text{colim}_\theta V \otimes n.$$ 

Then $A(V)$ is a functorial retract of $V \otimes n$. Since the map $\theta$ factors through $Q_{(2n-1)}(V) \otimes L_2(V) \otimes \cdots \otimes \alpha$, $A(V)$ is a functorial retract of $Q_{(2n-1)}(V) \otimes L_2(V) \otimes \cdots \otimes \alpha$. Now let $V$ be a 2-dimensional module with basis $\{u, v\}$. Then the element

$$v^{2n-1}[u, v]^{n-\alpha}$$

is a fixed point of the map $\theta$. It follows that $A$ is a nontrivial functor. Since $A(V)$ has one top degree element in degree $a|v| + (n-a)|u|$, the functor $Q_{(a, n-a)}$ is a retract of $A$ by Lemma 5.5. Thus $Q_{(a, n-a)}$ is a retract of the functor $Q_{(2n-1)} \otimes L_2 \otimes \cdots \otimes \alpha$ and hence the result. 

Using Theorems 4.6, 5.6 and 5.7, we have proved following theorem.

**Theorem 5.8.** The decomposition in Proposition 4.4 is a complete functorial decomposition.

### 6. Other Applications

6.1. **Decomposition Numbers.** Let $V$ be an ungraded module over $\mathbb{Z}/p$. Let $\hat{V}$ be a free $\mathbb{Z}/p$-module such that $V = \hat{V} \otimes \mathbb{Z}/p$ and let $\tilde{V} = \hat{V} \otimes \mathbb{Q}$. Let $Q_\lambda$ be the primitive idempotent in $\mathbb{Z}/p(S_n)$ corresponding a $p$-regular partition $\lambda$. Then we have a $p$-modular system

$$Q_\lambda(V) \overset{\text{mod } p}{\longleftrightarrow} \tilde{Q}_\lambda(\hat{V}) \longrightarrow \tilde{Q}_\lambda(V).$$

Then $\tilde{Q}_\lambda(\hat{V})$ is a module over $U^\hat{V}$ and so there is a $U^\hat{V}$-isomorphism

$$\tilde{Q}_\lambda(\hat{V}) \cong \bigoplus_{\mu} W^\mu(\hat{V}) \otimes d_{\mu, \lambda},$$

where $W^\mu(\hat{V})$ is the Weyl module corresponding to a partition $\lambda$. Suppose that $\text{dim } V = k \leq n$. By Theorems 5.4 and 5.6, the number $d_{\mu, \lambda}$ is the entries in position $(\mu, \lambda)$ of the decomposition matrix when $\text{Len}(\lambda) \leq k$ and $\text{Len}(\mu) \leq k$ (See [11]).

Now we specialize at the case $p = 2$ and $\text{dim } V = 2$. We are able to compute $d_{\mu, \lambda}$ for $\text{Len}(\lambda) \leq 2$ and $\text{Len}(\mu) \leq 2$ using our explicit decomposition for $V \otimes n$. Let $\mu = (\mu_1, \mu_2)$ be a partition of $n$ and let $n/2 < a \leq n$. Let

$$\mu^a = \begin{cases} 0 & \text{if } \mu_1 > a \\ (\mu_1 - (n-a), \mu_2 - (n-a)) & \text{if } \mu_1 \leq a \end{cases}$$
be a partition of $2a - n$.

**Theorem 6.1.** Let $\mu = (\mu_1, \mu_2)$ be a partition of $n$. Then there is an formula

$$d_{(a,n-a)\mu} = d_{(2a-n)\mu^*}.$$  

**Proof.** By Theorem 5.7, there is a $\mathcal{U}^V$-isomorphism

$$Q_{(a,n-a)}(V) \cong Q_{2a-n}(V) \otimes L_2(V)^{\otimes n-a}.$$  

The assertion follows. \qed

Now we are going to determine $d_{(n)\mu}$ for $\text{Len}(\mu) \leq 2$.

**Lemma 6.2.** If $2^k \leq n$, then there is a short exact sequence of modules over $\mathcal{U}^V$

$$0 \to S_{n-2^k}(V) \otimes L_2(V)^{\otimes 2^k} \to S_n(V) \otimes P_{2^k}(V) \to S_{n+2^k}(V).$$  

**Proof.** Let $j: P_{2^k}(V) \to S_{2^k}(V)$ be the inclusion and let $q: S_n(V) \otimes P_{2^k}(V) \to S_{n+2^k}(V)$ be the composite

$$S_n(V) \otimes P_{2^k}(V) \overset{j}{\longrightarrow} S_n(V) \otimes S_{2^k}(V) \overset{\text{mult}}{\longrightarrow} S_{n+2^k}.$$  

Then $q$ is an epimorphism over $\mathcal{U}^V$. Let

$$g: S_{n-2^k}(V) \otimes L_2(V)^{\otimes 2^k} \to S_n(V) \otimes P_{2^k}(V)$$  

be the $k$-linear map defined by

$$g(x \otimes [u, v]^{2^k}) = xu^{2^k} \otimes v^{2^k} - xv^{2^k} \otimes u^{2^k}.$$  

Then $q \circ g = 0$ and so we have a $\mathcal{U}^V$-map $g: S_{n-2^k}(V) \otimes L_2(V)^{\otimes 2^k} \to \text{Ker}(q)$, which is an isomorphism. The assertion follows. \qed

Given a positive integer $n$, let

$$n = 2^k + 2^{k-1} + \cdots + 2 + 1 + 2^a + \cdots + 2^a,$$

where $0 \leq a_1 < a_2 < \cdots < a_s \leq k$.

**Theorem 6.3.** There is an isomorphism of modules over $\mathcal{U}^{\tilde{V}}$

$$\tilde{Q}_n(\tilde{V}) \cong \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_r \leq s} L_2(\tilde{V})^{2^{a_i} + \cdots + 2^{a_r}} \otimes W^{(n-2^{a_1} - \cdots - 2^{a_r})}(\tilde{V}).$$  

**Proof.** By Theorem 5.8, we have

$$Q_n(V) \cong P_{2^k}(V) \otimes P_{2^{k-1}}(V) \cdots P_1(V) \otimes P_{2^{a_1}}(V) \otimes \cdots \otimes P_{2^{a_s}}(V)$$

$$\cong S_{2^k + 1}(V) \otimes P_{2^{a_1}}(V) \otimes \cdots \otimes P_{2^{a_s}}(V).$$

The assertion follows by induction on $s$ and by using Lemma 6.2. \qed

This gives the following theorem.
Theorem 6.4. Let $\mu = (\mu_1, \mu_2)$ be a partition of $n$ and let
\[
    n = 2^k + 2^{k-1} + \cdots + 1 + 2^{a_1} + 2^{a_2} + \cdots + 2^{a_s}
\]
with $0 \leq a_1 < a_2 < \cdots < a_s \leq k$. Then
\[
d_{(n|\mu)} = \begin{cases} 
    1 & \text{if } \mu_2 = 2^{a_1} + \cdots + 2^{a_t} \text{ for some } 1 \leq i_1 < i_2 < \cdots < i_t \leq s \\
    0 & \text{otherwise}
\end{cases}
\]

6.2. Irreducible $GL(2; k)$-modules. Let $k$ be an infinite field of characteristic 2. Let $V$ be a two-dimensional module with a basis $u, v$. The irreducible $GL(2; k)$-modules are classified as the head of Weyl modules (See [9]). We are going to show that irreducible $GL(2; k)$-modules have tensor product decompositions. Let $(a, n-a)$ be a partition of $n$. According to the definition of the Weyl modules [11], we have
\[
    W^{(a, n-a)}(V) \cong L_2(V)^{\otimes a} \otimes W^{2a-n}(V).
\]
Since $L_2(V)$ is of dimension one, it suffices to look at the head of $W^{(n)}(V)$ for each $n$.

Theorem 6.5. Let $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_t}$ with $0 \leq a_1 < a_2 < \cdots < a_t$ be the 2-adic resolution. Then
\[
    \text{Hd}(W^{(n)}(V)) \cong P_{2^{a_1}}(V) \otimes \cdots \otimes P_{2^{a_t}}(V).
\]

Proof. According to [11, Theorem 8.3.1], $W^{(n)}(V)$ is a cyclic $A^V$-submodule of $V^{\otimes n}$ generated by the bottom cell $u^n$. Let $\phi : W^{(n)}(V) \rightarrow S_n(V)$ be the composite
\[
    W^{(n)}(V) \hookrightarrow T_n(V) = V^{\otimes n} \twoheadrightarrow S_n(V)
\]
and let $F(V)$ be the image of $\phi$ in $S_n(V)$. Let
\[
    \theta : P_{2^{a_1}}(V) \otimes \cdots \otimes P_{2^{a_t}}(V) \rightarrow S(V)
\]
be the $k$-linear map defined by
\[
    \theta(x_1 \otimes \cdots \otimes x_t) = x_1 \cdots x_t
\]
for $x_j \in P_{2^{a_j}}(V)$. It is routine to check that the image of $\theta$ lies in $F(V)$ and the map
\[
    \theta : P_{2^{a_1}}(V) \otimes \cdots \otimes P_{2^{a_t}}(V) \rightarrow F(V)
\]
is an isomorphism of modules over $U^V$. By looking at Steenrod operations, it is easy to show that
\[
    P_{2^{a_1}}(V) \otimes \cdots \otimes P_{2^{a_t}}(V)
\]
is an irreducible bi-module over the Steenrod algebra. Since $k$ is an infinite field, $
\bigotimes_{j=1}^{t} P_{2^{a_j}}(V)$ is an irreducible $GL(2, k)$-module. According to [11, Theorem 8.3.2], $W^{(n)}$ has a unique irreducible quotient. The assertion follows. \qed
6.3. Decompositions of Self Smashes of suspensions of $\mathbb{R}P^2 \wedge \mathbb{C}P^2$. Let $Y = \Sigma^k \mathbb{R}P^2 \wedge \mathbb{C}P^2$ with $k \geq 3$. The space $Y$ is important for $v_1$-periodic homotopy theory. By the following theorem, we can apply Theorem 1.2 to give splittings of self smashes of $Y$.

**Theorem 6.6.** [4, 18] There is a homotopy decomposition

$$(\Sigma \mathbb{R}P^2)^{(3)} \simeq \Sigma^3 \mathbb{R}P^2 \vee \Sigma^6 \mathbb{R}P^2 \vee \Sigma^3 \mathbb{R}P^2 \wedge \mathbb{C}P^2.$$ 

Let $P^n(2) = \Sigma^{n-2} \mathbb{R}P^2$ for $n \geq 2$.

**Theorem 6.7.** Let $\phi: \tilde{H}_*(Y^{(n)}) \to \tilde{H}_*(Y^{(n)})$ be an idempotent over the Steenrod algebra. Then there is a homotopy idempotent $f: Y^{(n)} \to Y^{(n)}$ such that $f_* = \phi$.

**Proof.** Since $Y$ is a retract of a suspension of $(P^3(2))^{(3)}$, so $Y^{(n)}$ is a retract of a suspension of $(P^3(2))^{(3n)}$. The assertion follows from Theorem 1.2. \hfill $\square$

Let $V = H_*(\mathbb{R}P^2)$.

**Corollary 6.8.** The homology of a complete homotopy decomposition of $Y^{(n)}$ is a suspension of the complete decomposition of

$$(P_2(V))^{\otimes n} \otimes V^{\otimes n}$$

over the Steenrod algebra.

Proposition 4.4 gives a way to compute the homology of each homotopy indecomposable retract of $Y^{(n)}$.

**References**


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