# Contents

1 **Introduction to Linear Programming Problem.**  
   1.1 General Linear Programming problems.  
   1.2 Formulation of LP problems.  
   1.3 Compact form and Standard form of a general linear programming problem.  
   1.4 Piecewise linear convex objective functions.  

2 **Geometry of Linear Programming.**  
   2.1 Extreme point, vertex and basic feasible solution.  
   2.2 Existence of extreme points.  
   2.3 Optimality at some extreme point.  

3 **Development of the Simplex Method.**  
   3.1 Constructing Basic Solution.  
   3.2 Optimality Conditions.  
   3.3 The Simplex Method.  

4 **Implementing the Simplex Method.**  
   4.1 The Revised Simplex Method.  
   4.2 Simplex Tableau Implementation.  
   4.3 Starting the Simplex Algorithms.  
   4.4 Special Cases in Simplex Method Application.  

5 **Duality Theory.**  
   5.1 The dual problem.  
   5.2 The duality theorem.  
   5.3 Economic interpretation of optimal dual variables.  
   2  
   8  
   13  
   17  
   20  
   28  
   30  
   31  
   32  
   38  
   45  
   49  
   53  
   59  
   75  
   80  
   81  
   84  
   92
5.4 The dual Simplex Method. ................................. 94

6  Sensitivity and Postoptimality Analysis. .................. 99
   6.1 A new variable is added. ............................... 100
   6.2 A new constraint is added. ............................ 102
   6.3 Changes in the requirement vector $b$. ............... 104
   6.4 Changes in the cost vector $c$. ....................... 105
   6.5 Changes in a nonbasic column of $A$. ............... 107
   6.6 Applications. ........................................ 108

7  Transportation Problems. .................................. 114
   7.1 Transportation Models and Tableaus. ................. 114
   7.2 The Transportation Algorithm ......................... 118
   7.3 Unbalanced Transportation model. ..................... 129
Chapter 1

Introduction to Linear Programming Problem.

Linear programming is a mathematical modeling technique designed to mini-
mize or minimize a linear cost function subject to a finite set of linear equality
and inequality constraints. It is the basis for the development of solution al-
gorithms of other (more complex) types of Operations research (OR) models,
including integer, nonlinear, and stochastic programming.

1.1 General Linear Programming problems.

In this section, the general linear programming problem is introduced fol-
lowed by some examples to help us familiarize with some basic terminology
used in LP.

Notation

1. For a matrix $A$, we denote its transpose by $A^T$.

2. An $n$-dimensional vector $x \in \mathbb{R}^n$ is denoted by a column vector

\[
    x = \begin{pmatrix}
        x_1 \\
        x_2 \\
        \vdots \\
        x_n
    \end{pmatrix}.
\]

3. For vectors $x = (x_1, x_2, \ldots, x_n)^T$ and
   $y = (y_1, y_2, \ldots, y_n)^T$, the following denotes the matrix multiplication:

\[
    x^T y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.
\]
In a general linear programming problem, a cost vector \( c = (c_1, c_2, \cdots, c_n)^T \) is given. The objective is to minimize or maximize a linear cost function

\[
c^T x = \sum_{i=1}^{n} c_i x_i
\]

over all vectors \( x = (x_1, x_2, \cdots, x_n)^T \), subject to a finite set of linear equality and inequality constraints. This can be summarized as follows:

Minimize \( c^T x \)

(Or maximize)

Subject to

\[
\begin{align*}
    a_i^T x &\geq b_i, & i &\in M_+, \\
    a_i^T x &\leq b_i , & i &\in M_-, \\
    a_i^T x &= b_i , & i &\in M_0, \\
    x_j &\geq 0 , & j &\in N_+, \\
    x_j &\leq 0 , & j &\in N_-
\end{align*}
\]

where \( a_i = (a_{i1}, a_{i2}, a_{i3}, \cdots, a_{in})^T \) is a vector in \( \mathbb{R}^n \) and \( b_i \) is a scalar.

Terminology

1. Variables \( x_i \) are called decision variables. There are \( n \) of them.

2. Each constraint is either an equation or an inequality of the form \( \leq \) or \( \geq \). Constraints of the form \( a_i^T x (\leq, =, \geq) b_i \) are sometimes known as functional constraints.

3. If \( j \) is in neither \( N_+ \) nor \( N_- \), there are no restrictions on the sign of \( x_j \). The variable \( x_j \) is said to be unrestricted in sign or a unrestricted variable.

4. A vector \( x = (x_1, x_2, \cdots, x_n)^T \) satisfying all of the constraints is called a feasible solution or feasible vector. The set of all feasible solutions is called the feasible set or feasible region.

5. The function \( c^T x \) is called the objective function or cost function.

6. A feasible solution \( x^* \) that minimizes (respectively maximizes) the objective function, i.e. \( c^T x^* \leq c^T x \) (respectively \( c^T x^* \geq c^T x \) ) for all feasible vectors \( x \) is called an optimal feasible solution or simply, an optimal solution. The value of \( c^T x^* \) is then called the optimal cost.

7. For a minimization (respectively maximization) problem, the cost is said to be unbounded or the optimal cost is \( -\infty \) (respectively the optimal cost is \( \infty \)) if for every real number \( K \) we can find a feasible solution \( x \) whose cost is less than \( K \) (respectively whose cost is greater than \( K \)).

8. Maximizing \( c^T x \) is equivalent to minimizing \( -c^T x \).
Example 1.1 (2-variables)
Consider the following LP problem:

\[
\begin{align*}
\text{minimize} & \quad -x_1 - x_2 \\
\text{subject to} & \quad x_1 + 2x_2 \leq 3 \\
& \quad 2x_1 + x_2 \leq 3 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

(a) Sketch the feasible region and find an optimal solution of the LP graphically.

(b) If the cost function is changed to \(-x_1 + 2x_2\), what is the optimal solution?

Observations

1. For any given scalar \(z\), the set of points \(x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\) such that \(c^T x = z\) is described by the line with equation \(z = -x_1 - x_2\). This line is perpendicular to the vector \(c = \begin{pmatrix} -1 \\ -1 \end{pmatrix}\). (WHY?)

2. Different values of \(z\) lead to different lines, parallel to each other. Sketch lines corresponding to \(z = 1\), and \(z = -1\).

3. Increasing \(z\) corresponds to moving the line \(z = -x_1 - x_2\) along the direction of the vector \(c = \begin{pmatrix} -1 \\ -1 \end{pmatrix}\). Thus, to minimize \(z\), the line is...
moved as much as possible in the direction of the vector \(-c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) (i.e. the opposite direction of the vector \(c = \begin{pmatrix} -1 \\ -1 \end{pmatrix}\)) within the feasible region.

4. The optimal solution \(x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) is a corner of the feasible set.

**Remark** In \(\mathbb{R}^2\), the equation \(a_i^T x = b_i\) describes a line perpendicular to \(a_i\), whereas in \(\mathbb{R}^3\), the equation \(a_i^T x = b_i\) describes a plane whose normal vector is \(a_i\). In \(\mathbb{R}^n\), the equation \(a_i^T x = b_i\) describes a hyperplane whose normal vector is \(a_i\). Moreover, \(a_i\) corresponds to the direction of increasing value of \(a_i^T x\).

**Example 1.2** (3-variable)
Consider the following LP problem:

\[
\begin{align*}
\text{minimize} & \quad -x_1 - x_2 - x_3 \\
\text{subject to} & \quad x_i \leq 1 \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

The feasible set is the unit cube, described by \(0 \leq x_i \leq 1, i = 1, 2, 3\), and \(c = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}\). Then the vector \(x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\) is an optimal solution.
Example 1.3 (4-variable)
Minimize $2x_1 - x_2 + 4x_3$
Subject to $x_1 + x_2 + x_4 \geq 2$
$3x_2 - x_3 = 5$
$x_3 + x_4 \leq 3$
$x_1 \geq 0$
$x_3 \leq 0$

Example 1.4 Consider the feasible set in $\mathbb{R}^2$ of the linear programming problem.

$$\text{minimize } c^T x$$
$$\text{Subject to } -x_1 + x_2 \leq 1$$
$$x_1 \geq 0$$
$$x_2 \geq 0.$$
(a) For the cost vector $\mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, there is a unique optimal solution $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(b) For $\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, there are multiple optimal solutions $\mathbf{x}$ of the form $\mathbf{x} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ where $0 \leq x_2 \leq 1$. The set of optimal solutions is bounded.

(c) For $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, there are multiple optimal solutions $\mathbf{x}$ of the form $\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ where $x_1 \geq 0$. The set of optimal solutions is unbounded (some $\mathbf{x}$ is of arbitrarily large magnitude).

(d) For $\mathbf{c} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, every feasible solution is not optimal. The optimal cost is unbounded or the optimal cost is $-\infty$. 
(e) Imposing additional constraint \( x_1 + x_2 \leq -2 \), there is no feasible solution.

This example illustrates the following possibilities for a Linear Programming problem.

(a) There is a unique optimal solution.
(b) There are multiple optimal solutions. The set of optimal solutions is bounded or unbounded.
(c) The optimal cost is \(-\infty\) and no feasible solution is optimal.
(d) The feasible set is empty. The problem is infeasible.

1.2 Formulation of LP problems.

The crux of formulating an LP model is:

**Step 1** Identify the unknown variables to be determined (decision variables) and represent them in terms of algebraic symbols.

**Step 2** Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of the decision variables.

**Step 3** Identify the objective or criterion and represent it as a linear function of the decision variables, which is to be maximized or minimized.

**Example 2.1 The diet problem**

Green Farm uses at least 800 kg of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

<table>
<thead>
<tr>
<th>Feedstuff</th>
<th>kg per kg of feedstuff</th>
<th>Cost ($ per kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Protein</td>
<td>Fiber</td>
</tr>
<tr>
<td>Corn</td>
<td>0.09</td>
<td>0.02</td>
</tr>
<tr>
<td>Soybean meal</td>
<td>0.60</td>
<td>0.06</td>
</tr>
</tbody>
</table>

The dietary requirements of the total feed stipulate at least 30% protein and at most 5% fiber. Green Farm wishes to determine the daily minimum-cost feed mix.

Formulate the problem as an LP problem.

**Solution**

Decision variables:

\[ x_1 = \text{kg of corn in the daily mix} \]
\( x_2 = \text{kg of soybean meal in the daily mix} \)

Constraints:
- Daily amount requirement: \( x_1 + x_2 \geq 800 \)
- Dietary requirements:
  - Protein: \( 0.09x_1 + 0.60x_2 \geq 0.3(x_1 + x_2) \)
  - Fiber: \( 0.60x_1 + 0.06x_2 \leq 0.05(x_1 + x_2) \)

Objective: minimize \( 0.3x_1 + 0.9x_2 \)

Thus, the complete model is

\[
\begin{align*}
\text{minimize} & \quad 0.3x_1 + 0.9x_2 \\
\text{subject to} & \quad x_1 + x_2 \geq 800 \\
& \quad -0.21x_1 + 0.3x_2 \geq 0 \\
& \quad 0.55x_1 + 0.01x_2 \geq 0 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Example 2.2 (The Reddy Mikks Company)
The Reddy Mikks Company owns a small paint factory that produces both interior and exterior house paints for wholesale distribution. Two basic raw materials, A and B, are used to manufacture the paints. The maximum availability of A is 6 tons a day; that of B is 8 tons a day. The daily requirement of the raw materials per ton of the interior and exterior paints are summarized in the following table:

<table>
<thead>
<tr>
<th>Raw Material</th>
<th>Exterior</th>
<th>Interior</th>
<th>Maximum Availability(tons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>1</td>
<td>8</td>
</tr>
</tbody>
</table>

A market survey has established that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. The survey also shows that the maximum demand for interior paint is limited to 2 tons daily. The wholesale price per ton is $3000 for exterior paint and $2000 for interior paint. How much interior and exterior paints should the company produce daily to maximize gross income?

Solution
Decision variables:
- \( x_1 = \text{number of tons of exterior paint produced daily} \)
- \( x_2 = \text{number of tons of interior paint produced daily} \)

Constraints:
- Use of material A daily: \( x_1 + 2x_2 \leq 6 \)
- Use of material B daily: \( 2x_1 + x_2 \leq 8 \)
- Daily Demand: \( x_2 \leq x_1 + 1 \)
Maximum Demand: $x_2 \leq 2$.
Objective: maximize $3000x_1 + 2000x_2$

Thus, the complete LP model is:

\[
\text{maximize} \quad 3000x_1 + 2000x_2 \\
\text{subject to} \quad x_1 + 2x_2 \leq 6 \\
\quad \quad \quad \quad 2x_1 + x_2 \leq 8 \\
\quad \quad \quad \quad -x_1 + x_2 \leq 1 \\
\quad \quad \quad \quad x_2 \leq 2 \\
\quad \quad \quad \quad x_1, x_2 \geq 0
\]

**Example 2.1** The diet problem

Suppose that there are $n$ different foods and $m$ different nutrients, and that we are given the following table with the nutritional content of a unit of each food:

<table>
<thead>
<tr>
<th>nutrient</th>
<th>food 1</th>
<th>\cdots</th>
<th>food n</th>
</tr>
</thead>
<tbody>
<tr>
<td>nutrient 1</td>
<td>$a_{11}$</td>
<td>\cdots</td>
<td>$a_{1n}$</td>
</tr>
<tr>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
</tr>
<tr>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
<td>\cdot</td>
</tr>
<tr>
<td>nutrient $m$</td>
<td>$a_{m1}$</td>
<td>\cdots</td>
<td>$a_{mn}$</td>
</tr>
</tbody>
</table>

Let $A$ be the $m \times n$ matrix with entries $a_{ij}$. The $j$th column $A_j$ represents the nutritional content of the $j$th food.

Let $b = (b_1, b_2, \ldots, b_m)^T$ be a vector with the requirements of an ideal diet, or equivalently, a specification of the nutritional contents of an ‘ideal food’.

Given the cost $c_j$ per unit of Food $j$, $j = 1, 2, \ldots, n$. The problem of mixing nonnegative quantities of available foods to synthesize the ideal food at minimal cost is an LP problem.

Let $x_j$, $j = 1, 2, \ldots, n$, be the quantity of Food $j$ to synthesize the ideal food. The formulation of the LP is as follows:

\[
\begin{align*}
\text{Minimize} \quad & c_1x_1 + c_2x_2 + \cdots + c_n x_n \\
\text{Subject to} \quad & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i, \quad i = 1, 2, \ldots, m, \\
\quad & x_j \geq 0, \quad j = 1, 2, \ldots, n.
\end{align*}
\]

A variant of this problem: Suppose $b$ specifies the minimal requirements of an adequate diet. Then $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$ is replaced by

\[
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i.
\]

**Example 2.2** A production problem

A firm produces $n$ different goods using $m$ different raw materials. Let $b_i$, $i = 1, 2, \ldots, m$, be the available amount of $i$th raw material.
The $j$th good, $j = 1, 2, \ldots, n$, requires $a_{ij}$ units of the $i$th raw material and results in a revenue of $c_j$ per unit produced. The firm faces the problem of deciding how much of each good to produce in order to maximize its total revenue.

Let $x_j$, $j = 1, 2, \ldots, n$, be the amount of the $j$th good. The LP formulation becomes:

Maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$, $i = 1, 2, \ldots, m,$

$x_j \geq 0$, $j = 1, 2, \ldots, n.$

**Example 2.3 Bank Loan Policy** (cf. Taha p. 39)

The ABC bank is in the process of formulating a loan policy involving a total of $12$ million. Being a full-service facility, the bank is obliged to grant loans to different clientele. The following table provides the types of loans, the interest rate charged by the bank, and the probability of bad debt from past experience:

<table>
<thead>
<tr>
<th>Type of loan</th>
<th>Interest rate</th>
<th>Probability of Bad Debt</th>
</tr>
</thead>
<tbody>
<tr>
<td>Personal</td>
<td>0.140</td>
<td>0.10</td>
</tr>
<tr>
<td>Car</td>
<td>0.130</td>
<td>0.07</td>
</tr>
<tr>
<td>Home</td>
<td>0.120</td>
<td>0.03</td>
</tr>
<tr>
<td>Farm</td>
<td>0.125</td>
<td>0.05</td>
</tr>
<tr>
<td>Commercial</td>
<td>0.100</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Bad debts are assumed unrecoverable and hence no interest revenue. Competition with other financial institutions in the area requires that the bank allocate at least 40% of the total funds to farm and commercial loans. To assist the housing industry in the region, home loans must equal at least 50% of the personal, car and home loans. The bank also has a stated policy specifying that the overall ratio for bad debts on all loans may not exceed 0.04. How should funds be allocated to these types of loans to maximize the net rate of return?

**Solution** Let $x_1, x_2, x_3, x_4, x_5$ (in million dollars) be the amount of funds allocated to Personal loan, Car loan, Home loan, Farm loan and Commercial loan respectively.

Net return:
- **Personal**: $0.9x_1(0.14) - 0.1x_1 = 0.026x_1$.
- **Car**: $0.93x_2(0.130) - 0.07x_2 = 0.0509x_2$.
- **Home**: $0.97x_3(0.120) - 0.03x_3 = 0.0864x_3$.
- **Farm**: $0.95x_4(0.125) - 0.05x_4 = 0.06875x_4$.
- **Commercial**: $0.98x_5(0.100) - 0.02x_5 = 0.078x_5$.

Total Fund:

$x_1 + x_2 + x_3 + x_4 + x_5 \leq 12$
Competition:
\[
\frac{x_4 + x_5}{x_1 + x_2 + x_3 + x_4 + x_5} \geq 0.4
\]
\[\iff 0.4x_1 + 0.4x_2 + 0.4x_3 - 0.6x_4 + 0.6x_5 \leq 0\]

Housing industry:
\[x_3 \geq 0.5(x_1 + x_2 + x_3) \iff 0.5x_1 + 0.5x_2 - 0.5x_3 \leq 0\]

Overall bad debt:
\[
\frac{0.1x_1 + 0.07x_2 + 0.03x_3 + 0.05x_4 + 0.02x_5}{x_1 + x_2 + x_3 + x_4 + x_5} \leq 0.04
\]
\[\iff 0.06x_1 + 0.03x_2 - 0.01x_3 + 0.01x_4 - 0.02x_5 \leq 0\]

The LP formulation:

\[
\text{maximize} \quad 0.026x_1 + 0.0509x_2 + 0.0864x_3 + 0.06875x_4 + 0.078x_5
\]
\[
\text{subject to} \quad x_1 + x_2 + x_3 + x_4 + x_5 \leq 12
\]
\[
0.4x_1 + 0.4x_2 + 0.4x_3 - 0.6x_4 + 0.6x_5 \leq 0
\]
\[
0.5x_1 + 0.5x_2 - 0.5x_3 \leq 0
\]
\[
0.06x_1 + 0.03x_2 - 0.01x_3 + 0.01x_4 - 0.02x_5 \leq 0
\]
\[x_1, x_2, x_3, x_4, x_5 \geq 0\]

**Example 2.4** (Work Scheduling Problem)

A post office requires different numbers of full-time employees on different days of the weeks. The number of full-time employees required on each day is given below:

<table>
<thead>
<tr>
<th>Day</th>
<th>Number of Employees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 1</td>
<td>Monday 17</td>
</tr>
<tr>
<td>Day 2</td>
<td>Tuesday 13</td>
</tr>
<tr>
<td>Day 3</td>
<td>Wednesday 15</td>
</tr>
<tr>
<td>Day 4</td>
<td>Thursday 19</td>
</tr>
<tr>
<td>Day 5</td>
<td>Friday 14</td>
</tr>
<tr>
<td>Day 6</td>
<td>Saturday 16</td>
</tr>
<tr>
<td>Day 7</td>
<td>Sunday 11</td>
</tr>
</tbody>
</table>

Union rules state that each full-time employee must work five consecutive days and then receive two days off. The post office wants to meet its daily requirements with only full-time employees. Formulate an LP that the post office can use to minimize the number of full-time employees that must be
hired.

Let $x_j$ be the number of employees starting their week on Day $j$. The formulation of the LP becomes:

Minimize $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$
Subject to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 17$
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 13$
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 15$
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 19$
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 14$
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 16$
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 11$
$x_j \geq 0$, $x_j$ integer.

Note The additional constraint that $x_j$ must be an integer gives rise to a linear integer programming problem. Finding optimal solutions to general integer programming problems is typically difficult.

1.3 Compact form and Standard form of a general linear programming problem.

Compact form of a general linear programming problem

In a general linear programming problem, note that each linear constraint, be it an equation or inequality, can be expressed in the form $a_i^T x \geq b_i$.

1. $a_i^T x = b_i \iff a_i^T x \geq b_i$ and $a_i^T x \leq b_i$.

2. $a_i^T x \geq b_i \iff -a_i^T x \leq -b_i$.

3. Constraints $x_j \geq 0$ or $x_j \leq 0$ are special cases of constraints of the form $a_i^T x \geq b_i$, where $a_i$ is a unit vector and $b_i = 0$.

Thus, the feasible set in a general linear programming problem can be expressed exclusively in terms of inequality constraints of the form $a_i^T x \geq b_i$.

Suppose all linear constraints are of the form $a_i^T x \geq b_i$ and there are $m$ of them in total. We may index these constraints by $i = 1, 2, \ldots, m$.

Let $b = (b_1, b_2, \ldots, b_m)^T$, and $A$ be the $m \times n$ matrix whose rows are $a_1^T, a_2^T, \ldots, a_m^T$, i.e.

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}.$$
Then the constraints $a^T_i x \geq b_i$, $i = 1, 2, \ldots, m$, can be expressed compactly in the form $Ax \geq b$. ($Ax \geq b$ denotes for each $i$, the $i$ component of $Ax$ is greater than or equal to the $i$th component of $b$.)

The general linear programming problem can be written compactly as:

$$
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{(or maximize)} & \\
\text{subject to} & \quad Ax \geq b
\end{align*}
$$

A linear programming problem of this form is said to be in compact form.

**Example 3.1** Express the following LP problem in Example 1.3 in compact form.

Minimize $2x_1 - x_2 + 4x_3$

Subject to

$$
\begin{align*}
3x_2 & \geq 2 \\
3x_2 - x_3 & = 5 \\
x_3 + x_4 & \leq 3 \\
x_1 & \geq 0 \\
x_3 & \leq 0
\end{align*}
$$

Rewrite the above LP as

Minimize $2x_1 - x_2 + 4x_3$

Subject to

$$
\begin{align*}
2x_1 - x_2 + x_4 & \geq 2 \\
3x_2 - x_3 & \geq 5 \\
-3x_2 + x_3 & \geq -5 \\
-3x_3 & \geq -3 \\
x_1 & \geq 0 \\
x_3 & \geq 0
\end{align*}
$$

which is in the compact form with

$$
c = \begin{pmatrix} 2 \\ -1 \\ 4 \\ 0 \end{pmatrix}, \ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},
$$

$$
b = \begin{pmatrix} 2 \\ 5 \\ -5 \\ -3 \\ 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
$$

**Standard Form Linear Programming Problem**
A linear programming problem of the form

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{(or maximize)} & \quad \text{subject to} \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

is said to be in **standard form**.

**Note** Two optimization problems are said to be **equivalent** if an optimal solution to one problem can be constructed from an optimal solution to another.

A general linear programming problem can be transformed into an equivalent problem in standard form by performing the following steps when necessary:

1. **Elimination of nonpositive variable and free variables.**
   
   Replace nonpositive variable \( x_j \leq 0 \) by \( \bar{x}_j = -x_j \), where \( \bar{x}_j \geq 0 \).

   Replace unrestricted variable \( x_j \) by \( x^+_j - x^-_j \), and where new variables \( x^+_j \geq 0 \) and \( x^-_j \geq 0 \).

2. **Elimination of inequality constraints.**
   
   An inequality constraint \( \sum_{j=1}^{n} a_{ij}x_j \leq b_i \) can be converted to an equality constraint by introducing a **slack variable** \( s_i \) and the standard form constraints

   \[
   \sum_{j=1}^{n} a_{ij}x_j + s_i = b_i, \quad s_i \geq 0.
   \]

   For example, \( x_1 + 2x_2 \leq 3 \) is converted to \( x_1 + 2x_2 + S_1 = 3, \quad S_1 \geq 0 \).

   An inequality constraint \( \sum_{j=1}^{n} a_{ij}x_j \geq b_i \) can be converted to an equality constraint by introducing a **surplus variable** \( s_i \) and the standard form constraints

   \[
   \sum_{j=1}^{n} a_{ij}x_j - s_i = b_i, \quad s_i \geq 0.
   \]

   For example, \( 3x_1 + 4x_2 \geq 1 \) is converted to \( 3x_1 + 4x_2 - S_1 = 1, S_1 \geq 0 \).

**Example 3.2** Express the following LP problem in Example 1.3 in standard form.
Minimize \( 2x_1 - x_2 + 4x_3 \)
Subject to
\[
\begin{align*}
3x_2 - x_3 &= 5 \\
x_3 + x_4 &\geq 3 \\
3x_2 - x_3 &\geq 0 \\
x_3 &\leq 0
\end{align*}
\]
Replace \( x_2 = x_2^+ - x_2^- \), \( x_3 = -x_3' \), and \( x_4 = x_4^+ - x_4^- \).
Add a surplus variable \( S_1 \) to the \( \geq \)-constraint, and add a slack variable \( S_2 \) to the \( \leq \)-constraint.

Minimize \( 2x_1 - x_2^+ + x_2^- - 4x_3' \)
Subject to
\[
\begin{align*}
3x_2^+ - 3x_2^- + x_3' - x_3^- + x_4^+ - x_4^- - S_1 &= 2 \\
- x_3' + x_4^+ - x_4^- + S_2 &= 3 \\
x_1, x_2^+, x_2^-, x_3', x_4^+, x_4^-, S_1, S_2 &\geq 0
\end{align*}
\]
which is in the standard form with
\[
c = \left( \begin{array}{c}
2 \\
-1 \\
1 \\
-4 \\
0 \\
0 \\
0 \\
0
\end{array} \right), \quad x = \left( \begin{array}{c}
x_1 \\
x_2^+ \\
x_2^- \\
x_3' \\
x_3^- \\
x_4^+ \\
x_4^- \\
S_1 \\
S_2
\end{array} \right),
\]
\[
b = \left( \begin{array}{c}
2 \\
5 \\
3
\end{array} \right) \quad \text{and} \quad A = \left( \begin{array}{ccccccc}
1 & 1 & -1 & 0 & 1 & -1 & -1 & 0 \\
0 & 3 & -3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & -1 & 0 & 1
\end{array} \right).
\]

Remark (Why do we need different forms of LP problems?)

1. The general (compact) form \( Ax \geq b \) is often used to develop the theory of linear programming.
2. The standard form \( Ax = b, x \geq 0 \) is computationally convenient when it comes to algorithms such as simplex methods.

Visualizing standard form problems

Example 3.3. Consider the following linear programming problem:

\[
\begin{align*}
\text{minimize} & \quad -x_1 - x_2 \\
\text{subject to} & \quad x_1 + 2x_2 \leq 6 \\
& \quad 2x_1 + x_2 \leq 4 \\
& \quad x_1 - x_2 \leq 0 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]
Its equivalent standard LP is:

\[
\begin{align*}
\text{minimize} & \quad -x_1 - x_2 \\
\text{subject to} & \quad x_1 + 2x_2 + S_1 = 6 \\
& \quad 2x_1 + x_2 + S_2 = 4 \\
& \quad x_1 - x_2 + S_3 = 0 \\
& \quad x_1, x_2, S_1, S_2, S_3 \geq 0.
\end{align*}
\]

**Observation** There are 3 equations and 5 variables.

1. Points in the solution space can be represented in terms of \( x_1, x_2, S_1, S_2, S_3 \).
2. \( S_i = 0 \) or \( x_i = 0 \) \(\implies\) edge of feasible space.
3. \( S_i > 0 \) or \( x_i > 0 \) will move the feasible points from the edge towards the interior of the feasible space.
4. Two of the five variables equal to zero \(\implies\) a corner point.
5. The feasible space is a 2-dimensional set whose edge corresponds to one of \( x_1, x_2, S_1, S_2 \geq 0 \).

**Remark**
In general, the feasible set \( (Ax = b, \ x \geq 0, \text{ where } A \text{ is } m \times n, \ m \leq n ) \) of a standard form LP is an \((n - m)\)-dimensional set constrained by the linear inequalities \( x_i \geq 0, \ i = 1, 2, \cdots, n \).

### 1.4 Piecewise linear convex objective functions.

Piecewise linear convex function
The notation \( \max_{i=1,\ldots,m}\{a_i\} \) denotes the maximum value among \( a_1, a_2, \cdots, a_m \).
A function of the form $\max_{i=1,\ldots,m}(c_i^T x + d_i)$ is called a piecewise linear convex function.

**Example 4.1**
(a) Sketch the graph of $y = \max(2x, 1-x, 1+x)$.
(b) Express the absolute value function $f(x) = |x|$ as a piecewise linear convex function.

The following problem is not a formulation of an LP problem.

Minimize $\max(x_1, x_2, x_3)$
Subject to $2x_1 + 3x_2 \leq 5$
$x_2 - 2x_3 \leq 6$
$x_3 \leq 7$
$x_1, x_2, x_3 \geq 0$.

However, it can be converted to an equivalent LP problem by the next proposition.

**Proposition**
The minimization problem

\[
(I) \quad \text{minimize } \max_{i=1,\ldots,m}(c_i^T x + d_i) \\
\text{subject to } Ax \geq b.
\]

is equivalent to the linear programming problem

\[
(II) \quad \text{minimize } z \\
\text{subject to } z \geq c_i^T x + d_i, \quad i = 1, \ldots, m, \\
Ax \geq b.
\]

where the decision variables are $z$ and $x$.

**Proof**
Note: $\max_{i=1,\ldots,m}\{a_i\} = \min\{u \mid u \geq a_i, \ i = 1, 2, \ldots, m\}$, the smallest upper bound of the set $\{a_i \mid i = 1, 2, \ldots, m\}$.
Thus

\[
(I) \quad \text{minimize } \max_{i=1,\ldots,m}(c_i^T x + d_i) \\
\text{subject to } Ax \geq b.
\]

is equivalent to

\[
\text{minimize } \min\{z \mid z \geq (c_i^T x + d_i), \ i = 1, 2, \ldots, m\} \\
\text{subject to } Ax \geq b.
\]
which is in turn equivalent to

\[
\begin{align*}
\text{minimize} & \quad z \\
(II) \text{ subject to} & \quad z \geq c_i^T x + d_i, \ i = 1, 2, \ldots, m \\
& \quad Ax \geq b.
\end{align*}
\]

**Corollary**
The following maximization problems are equivalent:

\[
\begin{align*}
\text{maximize} & \quad \min_{i=1,\ldots,m} (c_i^T x + d_i) \\
(I') \text{ subject to} & \quad Ax \geq b.
\end{align*}
\]

\[
\begin{align*}
\text{maximize} & \quad z \\
(II') \text{ subject to} & \quad z \leq c_i^T x + d_i, \ i = 1, \ldots, m. \\
& \quad Ax \geq b.
\end{align*}
\]

**Example 4.2** Express the following as an LP problem.

Minimize \( \max (3x_1 - x_2, x_2 + 2x_3) \)
Subject to \( 2x_1 + 3x_2 \leq 5 \)
\( x_2 - 2x_3 \leq 6 \)
\( x_3 \leq 7 \)
\( x_1, x_2, x_3 \geq 0 \)

**Example 4.3**
A machine shop has one drill press and 5 milling machines, which are to be used to produce an assembly consisting of two parts, 1 and 2. The productivity of each machine for the two parts is given below:

<table>
<thead>
<tr>
<th>Part</th>
<th>Drill</th>
<th>Mill</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>15</td>
</tr>
</tbody>
</table>

It is desired to maintain a balanced loading on all machines such that no machine runs more than 30 minutes per day longer than any other machine (assume that the milling load is split evenly among all five milling machines). Assuming an 8-hour working day, formulate the problem as a linear programming model so as to obtain the maximum number of completed assemblies.
Chapter 2

Geometry of Linear Programming.

In this chapter, we consider the compact form of a general LP,

\[
\begin{align*}
\text{Minimize} & \quad c^T x \\ 
\text{Subject to} & \quad Ax \geq b.
\end{align*}
\]

We characterize corner points of the feasible set \( \{x | Ax \geq b\} \) geometrically (via extreme points and vertices) and algebraically (via basic feasible solution).

The main results state that a nonempty polyhedron has at least one corner point if and only if it does not contain a line, and if this is the case, the search for optimal solutions to linear programming problems can be restricted to corner points.

2.1 Extreme point, vertex and basic feasible solution.

A polyhedron or polyhedral set is a set that can be described in the form \( \{x \in \mathbb{R}^n | Ax \geq b\} \), where \( A \) is an \( m \times n \) matrix and \( b \) is a vector in \( \mathbb{R}^m \). Geometrically, a polyhedron is a finite intersection of half spaces \( a_i^T x \geq b_i \). The feasible set of a linear programming problem is a polyhedron.

Example 1.1

The feasible region of the following LP problem is a polyhedron.

\[
\begin{align*}
\text{minimize} & \quad -x_1 - x_2 \\ 
\text{subject to} & \quad -x_1 - 2x_2 \geq -3 \\ & \quad -2x_1 - x_2 \geq -3 \\ & \quad x_1, x_2 \geq 0.
\end{align*}
\]
Three Definitions of the concept of corner point.
Let $A$ be the $m \times n$ matrix with $i$th row $a_i^T = (a_{i1}, a_{i2}, \cdots, a_{in})$, $i = 1, 2, \cdots, m$ and $b = (b_1, b_2, \cdots, b_m)^T$. The polyhedron $P \subset \mathbb{R}^n$ defined by
$\{x \in \mathbb{R}^n \mid Ax \geq b\}$ can also be defined by the set of constraints $a_i^T x \geq b_i$, $i = 1, 2, \cdots, m$.

(a) A vector $x^* \in P$ is an extreme point of $P$ if we cannot find two vectors $y, z \in P$, and a scalar $\lambda \in (0, 1)$, such that $x^* = \lambda y + (1 - \lambda)z$.

(b) A vector $x^* \in P$ is a vertex of $P$ if we can find $v \in \mathbb{R}^n$ such that $v^T x^* < v^T y$ for all $y \in P - \{x^*\}$.

(c) A vector $x^* \in P$ is a basic feasible solution if we can find $n$ linearly independent vectors in the set $\{a_i \mid a_i^T x^* = b_i\}$. 
Definitions

1. If a vector $x^* \in \mathbb{R}^n$ satisfies $a_i^T x^* = b_i$ for some $i = 1, 2, \ldots, m$, the corresponding constraint $a_i^T x \geq b_i$ is said to be active (or binding) at $x^*$.

2. A vector $x^* \in \mathbb{R}^n$ is said to be of rank $k$ with respect to $P$, if the set $\{a_i \mid a_i^T x^* = b_i\}$ contains $k$, but not more than $k$, linearly independent vectors. In other words, the span of $\{a_i \mid a_i^T x^* = b_i\}$ has dimension $k$.

Thus, a vector $x^* \in P$ is a basic feasible solution if and only if it has rank $n$.

3. A vector $x^* \in \mathbb{R}^n$ (not necessary in $P$) is a basic solution if there are $n$ linearly independent vectors in the set $\{a_i \mid a_i^T x^* = b_i\}$. Moreover, every equality constraint (if any) must be satisfied at a basic solution.

4. Constraints $a_i^T x \geq b_i$, $i \in I$ are said to be linearly independent if the corresponding vectors $a_i$, $i \in I$, are linearly independent.

Example 1.2 Consider the following LP problem in Example 1.1:

\[
\begin{align*}
\text{minimize} & \quad -x_1 - x_2 \\
\text{subject to} & \quad x_1 + 2x_2 \leq 3 \\
& \quad 2x_1 + x_2 \leq 3 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

(a) The vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basic feasible solution.

(b) The vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a feasible solution with only one active constraint $x_1 = 0$. Thus, it has rank 1.

(c) The vector $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ is a feasible solution with no active constraint. Thus, it has rank 0.

(d) The vector $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is not a basic feasible solution. It is not feasible. Note that there are two linearly independent active constraints. Thus, it has rank 2. It is a basic solution.
Note
Given a finite number of linear inequality constraints, there can only be a finite number of basic solutions and hence a finite number of basic feasible solutions.

Example 1.3 Consider the polyhedron $P$ defined by

$$
\begin{align*}
x_1 + x_2 + x_4 &\geq 2 \\
3x_2 - x_3 &\geq 5 \\
x_3 + x_4 &\geq 3 \\
x_2 &\geq 0 \\
x_3 &\geq 0
\end{align*}
$$

Determine whether each of the following is a basic feasible solution.

(a) $x_a = (x_1, x_2, x_3, x_4)^T = (0, 2, 0, 3)^T$.
(b) $x_b = (x_1, x_2, x_3, x_4)^T = (0, 4, 7, -4)^T$.
(c) $x_c = (x_1, x_2, x_3, x_4)^T = (-8/3, 5/3, 0, 3)^T$.

Solution
Note $x \in \mathbb{R}^4$.

(a)

<table>
<thead>
<tr>
<th>constraint</th>
<th>satisfied?</th>
<th>active?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Yes, &gt;</td>
<td>No</td>
</tr>
<tr>
<td>2</td>
<td>Yes, &gt;</td>
<td>No</td>
</tr>
<tr>
<td>3</td>
<td>Yes, =</td>
<td>Yes</td>
</tr>
<tr>
<td>4</td>
<td>Yes, &gt;</td>
<td>No</td>
</tr>
<tr>
<td>5</td>
<td>Yes, =</td>
<td>Yes</td>
</tr>
</tbody>
</table>

All constraints are satisfied at $x_a$, it is feasible with two active constraints. Rank cannot be 4. Therefore $x_a$ is not a basic feasible solution.

(b) The first constraint is not satisfied at $x_b$. Thus it is not a basic feasible solution.

(c) Check that all constraints are satisfied and active at $x_c$ (Exercise).

Rank at $x_c$:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 3 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0
\end{bmatrix} \rightarrow \cdots \rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 3
\end{bmatrix}
$$

Thus rank at $x_c$ is 4.

The vector $x_c$ is a basic feasible solution.
Lemma A
Let $P$ be a nonempty polyhedron defined by $\{x \mid a_i^T x \geq b_i, i = 1, 2, \cdots, m\}$. Let $x^* \in P$ of rank $k$, $k < n$. Denote $I = \{i \mid a_i^T x^* = b_i\}$. Suppose there exists a nonzero vector $d$ such that $a_i^T d = 0$ for every $i \in I$, and $a_j^T d \neq 0$ for some $j \notin I$.

Then there exists $\lambda_0 > 0$ such that $x^* + \lambda d \in P$ for every $\lambda \in [-\lambda_0, \lambda_0]$.
Moreover, there exists $\lambda_*$ such that $x^* + \lambda_* d \in P$ with rank at least $k + 1$.

Remark A non-zero vector $d$ such that $x^* + \lambda d \in P$ for some $\lambda > 0$ is said to be a feasible direction.

*Proof.

How to find a suitable $\lambda_0 > 0$ such that the conclusion of the lemma holds?

Note that: $x^* + \lambda d \in P \iff a_j^T (x^* + \lambda d) \geq b_j \forall j \iff a_j^T x^* + \lambda a_j^T d \geq b_j \forall j$.

Denote $a_j^T x^* + \lambda a_j^T d \geq b_j$ by (*).

If $a_j^T d = 0$, then (*) holds since $a_j^T x^* + \lambda a_j^T d = a_j^T x^* \geq b_j$ for $\lambda \in \mathbb{R}$.

If $a_j^T d > 0$, then (*) holds whenever $\frac{a_j^T x^* - b_j}{-a_j^T d} \leq \lambda \leq \frac{a_j^T x^* - b_j}{|a_j^T d|}$.

If $a_j^T d < 0$, then (*) holds whenever $\frac{a_j^T x^* - b_j}{-a_j^T d} \geq \lambda \geq \frac{a_j^T x^* - b_j}{|a_j^T d|}$.

Thus, for $a_j^T x^* + \lambda a_j^T d \geq b_j \forall j$, we must have

$$\frac{a_j^T x^* - b_j}{-|a_j^T d|} \leq \lambda \leq \frac{a_j^T x^* - b_j}{|a_j^T d|} \quad \text{whenever } a_j^T d \neq 0.$$ 

Therefore we choose

$$\lambda_0 = \min\{\frac{a_j^T x^* - b_j}{|a_j^T d|} \mid a_j^T d \neq 0\}.$$ 

For $-\lambda_0 \leq \lambda \leq \lambda_0$, $a_j^T (x^* + \lambda d) \geq b_j$, for every $j = 1, 2, \cdots, m$.
Hence, $x^* + \lambda d \in P$. 

24
To prove the last part of the lemma.

The set \{j | a_j^T d \neq 0\} is finite, thus, \( \lambda_0 = \frac{a_j^T x^* - b_j}{|a_j^T d|} \), for some \( j_* \not\in I \). Let

\[ \lambda_* = \begin{cases} 
\lambda_0 & \text{if } a_j^T d < 0 \\
-\lambda_0 & \text{if } a_j^T d > 0.
\end{cases} \]

and \( \hat{x} = x^* + \lambda_* d \). Then \( a_j^T \hat{x} = b_j \) and \( a_i^T \hat{x} = a_i^T (x^* + \lambda_* d) = b_i \), for every \( i \in I \).

Since \( a_i^T d = 0 \), for all \( i \in I \), and \( d \neq 0 \), \( a_j^* \) is not a linear combination of \( a_i, i \in I \).

Therefore, the set \{a_j \mid a_j^T \hat{x} = b_j\} contains at least \( k + 1 \) linearly independent vectors.

Hence, \( \hat{x} \) has rank \( \geq k + 1 \). QED.
Remarks
The two geometric definitions, extreme point and vertex, are not easy to work with from the algorithmic point of view. It is desirable to have an algebraic definition, basic feasible solution, that relies on a representation of a polyhedron in terms of linear constraints and which reduces to an algebraic test.

The three definitions namely extreme point, vertex and basic feasible solution, are equivalent as proven in the next theorem. Therefore the three terms can be used interchangeably.

Theorem 1
Let $P$ be a nonempty polyhedron and let $x^* \in P$. Then the following are equivalent:

(a) $x^*$ is a vertex;
(b) $x^*$ is an extreme point;
(c) $x^*$ is a basic feasible solution.

*Proof. (We shall prove (a) $\implies$ (b) $\implies$ (c) $\implies$ (a).)

(a) $\implies$ (b): Vertex $\implies$ Extreme point. (We prove this by contradiction.)

Suppose $x^*$ is a vertex. Then there exists $v \in \mathbb{R}^n$ such that $v^T x^* < v^T y$ for every $y \in P - \{x^*\}$.

Suppose on the contrary that $x^*$ is not an extreme point. Then there exist two vectors $y_0, z_0 \in P$ and a scalar $\lambda \in (0, 1)$, such that $x^* = \lambda y_0 + (1 - \lambda)z_0$.

However, we have $v^T x^* < v^T y_0$ and $v^T x^* < v^T z_0$. Thus,

$v^T x^* = v^T (\lambda y_0 + (1 - \lambda)z_0)
= \lambda v^T y_0 + (1 - \lambda) v^T z_0
> \lambda v^T x^* + (1 - \lambda) v^T x^* = v^T x^*$

which gives rise to a contradiction. Thus, $x^*$ is an extreme point.
(b) $\implies$ (c): Extreme point $\implies$ basic feasible solution.

(We shall prove the contrapositive statement:
not basic feasible solution $\implies$ not extreme point.)

Suppose $x^* \in P$ is not a basic feasible solution.
Then the rank of $x^*$ is $k$, $k < n$.

(To show that $x^*$ is not an extreme point, we shall construct two vectors $y_0, z_0 \in P$ such that $x^* = \lambda y_0 + (1 - \lambda)z_0$ for some $\lambda \in (0, 1)$.)

Let $I = \{i|a_i^T x^* = b_i\}$. The set $\{a_i|a_i^T x^* = b_i\}$ has $k$ linearly independent vectors ($k < n$). Hence the linear system of equations $a_i^T x = 0$, $i \in I$, has infinitely many solutions. Choose a nonzero solution $d$, i.e. $a_i^T d = 0$, for $i \in I$.

If $a_i^T d = 0$ for every $j \not\in I$, then $a_i^T d = 0$, for every $i = 1, 2, \ldots, m$. Thus, we let $y_0 = x^* + d$ and $z_0 = x^* - d$. Both $y_0$ and $z_0$ are in $P$. (Exercise.)

If $a_i^T d \neq 0$ for some $j \not\in I$, then, by Lemma A, we can find $\lambda_0 > 0$ such that $x^* + \lambda_0 d \in P$ and $x^* - \lambda_0 d \in P$. Thus, we let $y_0 = x^* + \lambda_0 d$ and $z_0 = x^* - \lambda_0 d$.

Note that $x^* = \frac{1}{2}y_0 + \frac{1}{2}z_0$, i.e. $x^*$ is not an extreme point.

(c) $\implies$ (a): Basic feasible solution $\implies$ vertex. (We prove this directly.)

Suppose $x^*$ be a basic feasible solution. Let $I = \{i|a_i^T x^* = b_i\}$. The set $\{a_i|a_i^T x^* = b_i\}$ has $n$ linearly independent vectors. Hence the linear system of equations $a_i^T x = b_i$, $i \in I$, has a unique solution which is $x^*$.

We form a vector $v = \sum_{i \in I} a_i$, and shall prove that $v^T x^* < v^T y$ for $y \in P - \{x^*\}$.

Let $y \in P - \{x^*\}$. Then $a_i^T y \geq b_i, i = 1, 2, \ldots, m$ and hence
\[
 v^T y = \sum_{i \in I} a_i^T y \geq \sum_{i \in I} b_i = \sum_{i \in I} a_i^T x^* = v^T x^*.
\]

If $v^T y = v^T x^* = \sum_{i \in I} b_i$, then we must have $a_i^T y = b_i, i \in I$ because $a_i^T y \geq b_i$, for each $i$. Thus $y$ is a solution to the linear system $a_i^T x = b_i, i \in I$.

From the uniqueness of the solution, we must have $y = x^*$, contradicting $y \in P - \{x^*\}$.

Therefore, $v^T y > v^T x^*$ and this proves that $x^*$ is a vertex. QED
2.2 Existence of extreme points.

Geometrically, a polyhedron containing an infinite line does not contain an extreme point. As an example, the polyhedron \( P = \{ \begin{pmatrix} x \\ 0 \end{pmatrix} | x \in \mathbb{R} \} \subset \mathbb{R}^2 \) does not have an extreme point. In \( \mathbb{R}^3 \), \( x^* + \lambda d, \lambda \in \mathbb{R} \) describes a line which is parallel to \( d \) and passes through \( x^* \).

A polyhedron \( P \subset \mathbb{R}^n \) contains a line if there exists a vector \( x^* \in P \) and a nonzero vector \( d \in \mathbb{R}^n \) such that \( x^* + \lambda d \in P \) for all \( \lambda \in \mathbb{R} \).

**Theorem 2**

Suppose that the polyhedron \( P = \{ x \in \mathbb{R}^n | Ax \geq b \} \) is nonempty. Then \( P \) does not contain a line if and only if \( P \) has a basic feasible solution.

*Proof.*

\((\Rightarrow)\) Suppose \( P \) does not contain a line.

(Our aim is to show there is a basic feasible solution.)

Since \( P \) is nonempty, we may choose some \( x_0 \in P \).

**Case** rank of \( x_0 = n \).

Then \( x_0 \) is a basic feasible solution.

**Case** rank of \( x_0 = k < n \).

Let \( I = \{ i | a_i^T x_0 = b_i \} \). The set \( \{ a_i | a_i^T x_0 = b_i \} \) contains \( k \), but not more than \( k \), linearly independent vectors, where \( k < n \). The linear system of equations \( a_i^T x = 0, i \in I \), has infinitely many solutions. Choose a nonzero solution \( d \), i.e. \( a_i^T d = 0 \), for \( i \in I \).

**Claim:** \( a_j^T d \neq 0 \) for some \( j \notin I \).

**Proof.** Suppose \( a_i^T d = 0 \) \( \forall j \notin I \), Then \( a_i^T d = 0 \) for every \( i = 1, 2, \ldots, m \). For \( \lambda \in \mathbb{R} \), note that \( a_i^T(x_0 + \lambda d) = a_i^T x_0 \geq b_i \).

Therefore, we have \( x_0 + \lambda d \in P \), i.e. \( P \) contains the line \( x_0 + \lambda d \), a contradiction.

Thus, \( a_j^T d \neq 0 \) for some \( j \notin I \).

By Lemma A, we can find \( x_1 = x_0 + \lambda_* d \in P \) and the rank of \( x_1 \) is at least \( k + 1 \).

By repeating the same argument to \( x_1 \) and so on, as many times as needed, we will obtain a point \( x^* \) with rank \( n \), i.e. \( \{ a_i | a_i^T x^* = b_i \} \) contains \( n \) linearly independent vectors. Thus, there is at least one basic feasible solution.
Suppose $P$ has a basic feasible solution $x^*$. Then there exist $n$ linearly independent row vectors, say $a^T_1, a^T_2, \cdots, a^T_n$ of $A$ such that $a^T_i x^* = b_i$, $i = 1, 2, \cdots, n$.

Suppose, on the contrary, that $P$ contains a line, say $\hat{x} + \lambda d$, where $d \neq 0$.

Then, $a^T_i d \neq 0$ for some $i = 1, 2, \cdots, n$. (If not, $a^T_i d = 0$ for all $i = 1, 2, \cdots, n$, and hence $d = 0$, since $a^T_i, i = 1, 2, \cdots, n$, are linearly independent.)

Without loss of generality, we may assume $a^T_1 d \neq 0$.
Replacing $d$ by $-d$ if necessary, we may further assume $a^T_1 d > 0$.

However, $\hat{x} + \lambda d \not\in P$ for $\lambda < \frac{b_1 - a^T_1 \hat{x}}{a^T_1 d}$, since $a^T_1 (\hat{x} + \lambda d) < b_1$.
This contradicts the assumption that $P$ contains the line $\hat{x} + \lambda d$. (QED)

**Example 2.1** The polyhedron $P$ defined by
\[
\begin{align*}
x_1 &+ x_2 &+ x_4 &\geq 2 \\
3x_2 &- x_3 &+ x_4 &\geq 5 \\
x_3 &+ x_4 &\geq 3 \\
x_2 &\geq 0 \\
x_3 &\geq 0
\end{align*}
\]
contains a basic feasible solution, namely, $x^* = \left(\begin{array}{c}
-\frac{8}{3} \\
0 \\
3
\end{array}\right)$ (see Example 1.3).
Thus, by Theorem 2, $P$ does not contain a line.

A polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$ is said to be **bounded** if there exists a positive number $K$ such that $|x_i| \leq K$ for all $x = (x_1, x_2, \cdots, x_n)^T \in P$.

**Remark** A nonempty polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$ that does not contain a line must have a basic feasible solution. Thus, each of the following polyhedra has at least one basic feasible solution.

1. A nonempty bounded polyhedron.
2. A positive orthant $\{x \mid x \geq 0\}$.
2.3 Optimality at some extreme point.

Geometrically, if an LP problem has a corner point and an optimal solution, then an optimal solution occurs at some corner point. The next theorem justifies this geometrical insight. So, in searching for optimal solutions, it suffices to check on all corner points.

Theorem 3 Consider the linear programming problem of minimizing $c^T x$ over a polyhedron $P$. Suppose that $P$ has at least one extreme point and that there exists an optimal solution. Then there exists an optimal solution which is an extreme point of $P$.

Proof. We denote the optimal cost by $v$.

Let $Q = \{x \in P | c^T x = v\}$. Then $Q$ is a polyhedron.

Step 1 $Q$ has an extreme point $x^*$.

Since $P$ has at least one extreme point, $P$ does not contain a line, by Theorem 2. Hence $Q$, being a subset of $P$, does not contain a line. By Theorem 2, $Q$ has an extreme point, say $x^*$.

Step 2 $x^*$ is also an extreme point of $P$.

Suppose $x^*$ is not an extreme point of $P$.

Then there exists $\lambda \in (0, 1)$ and $y, z \in P$ such that $x^* = \lambda y + (1 - \lambda) z$.

Suppose either $c^T y > v$ or $c^T z > v$.

Then, we have $c^T x^* = c^T (\lambda y + (1 - \lambda) z) > v$, contradicting $c^T x^* = v$.

Therefore, both $c^T y = v$ and $c^T z = v$; thus, $y, z \in Q$ and $x^* = \lambda y + (1 - \lambda) z$. This contradicts $x^*$ being an extreme point of $Q$.

Thus, $x^*$ is an extreme point of $P$ and it is optimal (since $c^T x^* = v$). QED.

Corollary Consider the linear programming problem of minimizing $c^T x$ over a polyhedron $P$. Suppose that $P$ has at least one extreme point. Then, either the optimal cost is equal to $-\infty$, or there exists an extreme point which is optimal.

A nonempty polyhedron with an extreme point needs not have an optimal solution. However, if it is known that there is an optimal solution, then we can find an optimal solution within the set of extreme points.
Chapter 3
Development of the Simplex Method.

Throughout this chapter, we consider the standard form of a LP,

\[
\begin{align*}
\text{Minimize} & \quad c^T x \\
\text{Maximize} & \quad c^T x \\
\text{Subject to} & \quad Ax = b \\
& \quad x \geq 0.
\end{align*}
\]

Assume \( A \) is an \( m \times n \) matrix and rank \((A) = m\). Thus, row vectors \( a_i^T, i = 1, 2, \ldots, m, \) of \( A \) are linearly independent and \( m \leq n \). The \( i \)th column of \( A \) is denoted by \( A_i \).

Let \( P = \{ x \in \mathbb{R}^n \mid Ax = b, x \geq 0 \} \). Note that if \( P \neq \phi \), then \( P \) has an extreme point since it does not contain a line. Therefore, either the optimal value is unbounded or there exists an optimal solution which can be found among the finite set of extreme points.

Recall from Chapter 2, the following definition of a basic solution.

A vector \( x^* \in \mathbb{R}^n \) (not necessary in \( P \)) is a \textbf{basic solution} if there are \( n \) linearly independent vectors in the set \( \{ a_i \mid a_i^T x^* = b_i \} \). Moreover, every equality constraint (if any) must be satisfied at a basic solution.

Extreme points, equivalently basic feasible solutions, are basic solutions which are in \( P \). In this chapter, we develop a procedure to construct basic solutions. If the basic solution satisfies the nonnegativity constraint, we obtain a basic feasible solution. We also derive optimality conditions to check for optimality. Based on the procedure for obtaining the next basic solution and the optimality conditions, the simplex method is developed and a typical iteration is demonstrated.
3.1 Constructing Basic Solution.

Suppose $x^*$ is a basic solution of the standard form LP. Then $Ax^* = b$ and there are $n$ linearly independent active constraints at $x^*$. Since $A$ has rank $m$, there are $n - m$ linearly independent active constraints from $x \geq 0$. Therefore we have $n - m$ zero variables $x^*_i = 0$, where $x^*_i$ is the $i$-component of $x^*$. So, there are indices $B(1), B(2), \cdots, B(m)$ such that

$$x^*_i = 0 \text{ for } i \neq B(1), B(2), \cdots, B(m).$$

and

$$\sum_{i=1}^{m} A_{B(i)} x^*_{B(i)} = b.$$

The first theorem is a useful characterization of a basic solution. It allows us to construct a basic solution in a systematic way.

**Theorem**

Consider the constraints $Ax = b$ and $x \geq 0$ and assume that the $m \times n$ matrix $A$ has linearly independent rows. A vector $x^* \in \mathbb{R}^n$ is a basic solution if and only if $Ax^* = b$ and there exist indices $B(1), B(2), \cdots, B(m)$ such that

(a) The columns $A_{B(1)}, A_{B(2)}, \cdots, A_{B(m)}$ are linearly independent; and

(b) $x_i = 0$ for $i \neq B(1), B(2), \cdots, B(m)$.

**Proof.**

($\Leftarrow$) Suppose $x^* \in \mathbb{R}^n$ satisfies $Ax^* = b$ and there exist indices $B(1), B(2), \cdots, B(m)$ such that (a) and (b) are satisfied.

**Aim:** To show that there are $n$ linearly independent active constraints from:

$$\begin{cases} 
  Ax = b \\
  x_i = 0 \text{ for } i \neq B(1), B(2), \cdots, B(m),
\end{cases}$$

(1) and (2) imply that

$$\sum_{i=1}^{m} A_{B(i)} x^*_{B(i)} = \sum_{i=1}^{n} A_i x_i = Ax = b. \quad (3)$$

(3) is a linear system of equations in $m$ variables $x_{B(i)}$. Since $A_{B(1)}, A_{B(2)}, \cdots, A_{B(m)}$ are linearly independent, the linear system (3) has a unique solution and $x^*_{B(i)}$, $i = 1, 2, \cdots, m$, are uniquely determined. Hence, $x^*$ is uniquely determined.

Thus, $x^*$ is the unique solution to (1) and (2). Hence there are $n$ linearly independent active constraints from (1) and (2). We thus conclude that $x^*$ is a basic solution.
(⇒) Suppose \( x^* \) is a basic solution. By the definition of a basic solution, all equality constraints must be satisfied, thus, we have \( Ax^* = b \).

There are \( n \) linearly independent active constraints at \( x^* \) from constraints \( Ax = b \) and \( x \geq 0 \).

Since the rank of \( A \) is \( m \), there are \( n - m \) linearly independent active constraints from \( x \geq 0 \). So, there are indices \( B(1), B(2), \ldots, B(m) \) such that \( x^*_i \), the \( i \)-component of \( x^* \) satisfy

\[
x^*_i = 0 \text{ for } i \neq B(1), B(2), \ldots, B(m).
\]

Therefore, \( x^* \) is a solution of the linear system :

\[
\begin{align*}
Ax &= b, \\
x_i &= 0, i \neq B(1), B(2), \ldots, B(m).
\end{align*}
\]

Since there are \( n \) variables and there are \( n \) linearly independent active constraints, the solution to the linear system (***) is unique. Thus, the columns of coefficients of \( x_{B(i)} \) obtained from the linear system (***) are linearly independent. Hence, columns vectors \( A_{B(1)}, A_{B(2)}, \ldots, A_{B(m)} \) are linearly independent. QED.
**Terminology** Suppose $x$ is a basic solution.

1. Variables $x_{B(1)}, x_{B(2)}, \cdots, x_{B(m)}$ are called **basic variables**.

2. Variables $x_i = 0$ for $i \neq B(1), B(2), \cdots, B(m)$, are called **nonbasic variables**.

3. An $m \times m$ matrix $B$ obtained by arranging the $m$ basic columns next to each other is called a **basis matrix**. A vector $x_B$ can also be defined with the values of the basic variables. Thus,

$$B = \begin{bmatrix} A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)} \end{bmatrix} \quad \text{and} \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ \vdots \\ x_{B(m)} \end{bmatrix}.$$  

Note that $B$ is invertible and $Bx_B = b$ so that $x_B$ is the unique solution given by

$$x_B = B^{-1}b.$$  

4. The set of $m$ linearly independent columns $\{A_{B(1)}, A_{B(2)} \cdots A_{B(m)}\}$ is called a **basis**.

From the last theorem, all basic solutions to a standard form polyhedron can be constructed according to the following procedure.

**Procedure for constructing basic solution.**

1. Choose $m$ linearly independent columns

$$A_{B(1)}, A_{B(2)}, \cdots, A_{B(m)}.$$  

2. Let $x_i = 0$ for $i \neq B(1), B(2), \cdots, B(m)$.

3. Solve the system of $m$ linear equations $Ax = b$ for the unknowns $x_{B(1)}, \cdots, x_{B(m)}$.

**Remark** A basic solution $x$ constructed according to the above procedure is a basic feasible solution if and only if $x \geq 0$.  

34
Example 1.1 For the following constraints

\[
\begin{bmatrix}
1 & 1 & 2 & 1 & 0 & 0 & 0 \\
0 & 1 & 6 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
= 
\begin{bmatrix}
8 \\
12 \\
4 \\
6
\end{bmatrix}.
\]

\[x \geq 0.\]

(a) Find the basic solution associated with linearly independent columns \(A_4, A_5, A_6, A_7\). Is it a basic feasible solution?

(b) Show that columns \(A_1, A_2, A_3, A_4\) are linearly independent. Find the basis matrix \(B\) and the associated basic solution. Is it feasible?

(c) Do columns \(A_2, A_3, A_4, A_5\) form a basis matrix? If so, what is the associated basic solution?

Solution

(a) Note that \(A_4, A_5, A_6, A_7\) are linearly independent. Thus, we may proceed to find the associated basic solution.

We have \(B = [A_4, A_5, A_6, A_7] = I_4\) which is called a basis matrix.

Corresponding basic variables: \(x_4, x_5, x_6, x_7\).

Non-basic variables: \(x_1 = 0, x_2 = 0, x_3 = 0\).

Setting \(x_1 = 0, x_2 = 0, x_3 = 0\) to solve for \(x_4, x_5, x_6, x_7\):

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_B
\end{bmatrix}
= 
\begin{bmatrix}
8 \\
12 \\
4 \\
6
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
x_B
\end{bmatrix}
= 
\begin{bmatrix}
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}.
\]

We have

\[
\begin{bmatrix}
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
= 
\begin{bmatrix}
8 \\
12 \\
4 \\
6
\end{bmatrix} \geq 0. \text{ (Feasible)}
\]

Thus we obtain a basic feasible solution, namely,

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
8 \\
12 \\
4 \\
6
\end{bmatrix} \geq 0.
\]
(b) Check \( \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4 \) are linearly independent:

\[
\begin{bmatrix} \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow \cdots \rightarrow \mathbf{I}_4.
\]

Thus, \( \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4 \) are linearly independent and

\[
\mathbf{B} = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\]

is a basis matrix.

Nonbasic variables: \( x_5 = 0, x_6 = 0, x_7 = 0 \).

To find values of basic variables \( x_1, x_2, x_3, x_4 \):

\[
\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}_B = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}
\]

where \( \mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \).

Solving yields: \( \mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ -6 \end{bmatrix} \).

Thus, the associated basic solution is

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ -6 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\]

Since \( x_4 < 0 \), the basic solution is not feasible.

(c) Check for linear independence of \( \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5 \):

\[
\begin{bmatrix} \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 6 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\]

Columns \( \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5 \) are not linearly independent (WHY?). Thus, they do not form a basis matrix. (No need to proceed to find solution.)

**Exercise** Show \( \mathbf{B} = [\mathbf{A}_3, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7] \) is a basis matrix, and \( \mathbf{x}_B = (4, -12, 4, 6)^T \).
Adjacency and degeneracy.

Geometrically, adjacent basic feasible solutions are extreme points which are adjacent. The simplex method attempts to find an adjacent basic feasible solution that will improve the objective value.

Definition

Two distinct basic solutions to a set of linear constraints in \( \mathbb{R}^n \) are said to be adjacent if and only if the corresponding bases share all but one basic column, i.e. there are \( n - 1 \) linearly independent constraints that are active at both of them.

Example 1.2 Refer to the constraints in Example 1.1.

<table>
<thead>
<tr>
<th>Basic solution</th>
<th>Basic columns</th>
<th>Basic variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 8, 12, 4, 6)^T)</td>
<td>(A_1, A_5, A_6, A_7)</td>
<td>(x_1, x_5, x_6, x_7)</td>
</tr>
<tr>
<td>((0, 4, 0, -12, 4, 6)^T)</td>
<td>(A_3, A_5, A_6, A_7)</td>
<td>(x_3, x_5, x_6, x_7)</td>
</tr>
</tbody>
</table>

The above basic solutions are adjacent.

Note that, in standard form, the set of basic variables of two adjacent basic solutions differ by one component.

Definition

A basic solution \( \mathbf{x} \in \mathbb{R}^n \) is said to be degenerate if it has more than \( n \) active constraints, i.e. the number of active constraints at \( \mathbf{x} \) is greater than the dimension of \( \mathbf{x} \).

Geometrically, a degenerate basic solution is determined by more than \( n \) active constraints (overdetermined). In \( \mathbb{R}^2 \), a degenerate basic solution is at the intersection of 3 or more lines; whereas in \( \mathbb{R}^3 \), a degenerate basic solution is at the intersection of 4 or more planes.

In standard form, a basic solution \( \mathbf{x} \) is degenerate if some basic variable \( x_{B(i)} = 0 \), i.e. more than \( n - m \) components of \( \mathbf{x} \) are zero.

Example 1.3 For the following constraints

\[
\begin{bmatrix}
1 & 1 & 2 & 1 & 0 & 0 & 0 \\
0 & 1 & 3 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
8 \\
12 \\
4 \\
6
\end{bmatrix}
\]

\( \mathbf{x} \geq 0. \)

The basic feasible solution \( \mathbf{x} = (0, 0, 4, 0, 0, 0, 6)^T \), associated with basis \( \mathbf{B} = [ \mathbf{A_3} \mathbf{A_4} \mathbf{A_5} \mathbf{A_7} ] \) is degenerate and there are 9 active constraints at \( \mathbf{x} \).
3.2 Optimality Conditions.

In this section, we obtain optimality conditions to check whether a basic feasible solution is optimal. This is useful in the development of Simplex Method. The optimality conditions also provide a clue for searching a direction to improve the objective value in a neighbourhood of a basic feasible solution.

Let \( x^* \) be a basic feasible solution with the set \( \{B(1), \ldots, B(m)\} \) of basic indices, so that

\[
B = \begin{bmatrix}
A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)}
\end{bmatrix}
\]

and

\[
x^*_B = \begin{bmatrix}
x^*_B(1) \\
\vdots \\
x^*_B(m)
\end{bmatrix} = B^{-1}b \geq 0.
\]

When we move from \( x^* \) to an adjacent basic solution (may or may not be feasible) \( x' \), a nonbasic variable \( x_j \) of \( x^* \) becomes a basic variable of \( x' \).

There is an exchange of a basic variable and nonbasic variable. In the next lemma, we shall determine the feasible direction moving away from \( x^* \) so that the variable \( x_j \) becomes a basic variable.

**Lemma A**

Fix an index \( j \not\in \{B(1), \ldots, B(m)\} \). Suppose \( x' = x^* + \theta d \), where \( \theta > 0 \) and \( d = (d_1, d_2, \ldots, d_n)^T \) with \( d_j = 1 \) and \( d_i = 0 \), for every index \( i \not\in \{B(1), \ldots, B(m)\} \) and \( i \neq j \). Denote \( d_B = (d_{B(1)}, d_{B(2)}, \ldots, d_{B(m)})^T \). If \( x' \) is a feasible solution, then \( d_B = -B^{-1}A_j \) and \( B^{-1}b - \theta B^{-1}A_j \geq 0 \).

**Proof** In order to maintain feasibility of solution, we must have

\[
A(x') = b \quad \text{and} \quad x' \geq 0.
\]

i.e. \( A(x^* + \theta d) = b \) and \( x^* + \theta d \geq 0 \).

However, \( Ax^* = b \) and \( \theta > 0 \) so that \( A(x^* + \theta d) = b \) implies \( Ad = 0 \). Thus,

\[
0 = Ad = \sum_{i=1}^{n} A_i d_i = \sum_{i=1}^{m} A_{B(i)} d_{B(i)} + A_j = Bd_B + A_j.
\]

Therefore, \( Bd_B = -A_j \) and hence \( d_B = -B^{-1}A_j \).

Note that \( x^* + \theta d \geq 0 \) is equivalent to \( x_j^* + \theta d_i \geq 0 \ \forall \ i \).

Thus, \( x^*_B + \theta d_B \geq 0 \), i.e. \( B^{-1}b - \theta B^{-1}A_j \geq 0 \). [QED.]

In summary, we have obtained the vector \( d = (d_1, d_2, \ldots, d_n)^T \) where

\[
\begin{cases}
  d_j = 1, \\
  d_i = 0 \quad \text{for every non basic index} \ i \neq j, \ \text{and} \\
  d_B = -B^{-1}A_j.
\end{cases}
\]
Notes
1. If $i \not\in \{B(1), B(2), \ldots, B(m)\}$ and $i \neq j$, then the $i$-component of $x'$ is $x'_i = 0$ since $x^*_i = 0$, and $d_i = 0$. The $j$-component of $x'$ is $x'_j = \theta$ since $d_j = 1$.

2. The point $x'$ is obtained from $x^*$ by moving in the direction $d$. It is obtained from $x^*$ by selecting a nonbasic variable $x_j$ (i.e. $j \not\in \{B(1), \cdots, B(m)\}$) and increasing it to a positive value $\theta$, while keeping the remaining nonbasic variables $x_i$ at zero, i.e. $x' = x^* + \theta d$, where $d = (d_1, d_2, \cdots, d_n)^T$ with $d_j = 1$ and $d_i = 0$, for every nonbasic index $i, i \neq j$.

**Lemma B**
(a) If $B^{-1}A_j \leq 0$, then the polyhedron is unbounded in the $x_j$-direction.

(b) Suppose $x^*$ is nondegenerate, i.e. $x^*_B > 0$.
If $(B^{-1}A_j)_k > 0$ for some $k$, then $\theta \leq \frac{(B^{-1}b)_k}{(B^{-1}A_j)_k}$.

**Proof.**
(a) Since $x^*_B \geq 0$ and $B^{-1}A_j \leq 0$, we have from Lemma A, $x'_B = x^*_B - \theta B^{-1}A_j \geq 0$ whenever $\theta > 0$. Thus $x'_j = \theta$ is unbounded.

(b) Suppose $x^*$ is nondegenerate, i.e. $x^*_B > 0$.
If $(B^{-1}A_j)_k > 0$ for some component $k$, then $(B^{-1}b)_k - \theta (B^{-1}A_j)_k \geq 0$ yields
$$\theta \leq \frac{(B^{-1}b)_k}{(B^{-1}A_j)_k}.$$  

**Remark**
Suppose $(B^{-1}A_j)_k > 0$ for some $k$-th component. Let
$$\theta^* = \min\{\frac{(B^{-1}b)_k}{(B^{-1}A_j)_k} | (B^{-1}A_j)_k > 0\}.$$  

Then
$$\theta^* = \frac{(B^{-1}b)_l}{(B^{-1}A_j)_l},$$  

for some $l$. The feasible solution $x' = x^* + \theta^* d$ is a basic feasible solution which is adjacent to $x^*$, with associated basic variables
$$\{x_{B(1)}, \cdots, x_{B(l-1)}, x_{B(l+1)}, \cdots, x_{B(m)}, x_j\}.$$
Remark
If \( x^* \) is degenerate, i.e. there is some zero basic variable \( x_{B(k)} \), then the nonzero vector \( d \) may not be a feasible direction. This happens if a basic variable \( x_{B(k)} = 0 \) and \( d_{B(k)} < 0 \), so that \( x^* + \theta d \notin P \) for all positive scalar \( \theta \).

Example 2.1 Consider the LP problem

\[
\begin{align*}
\text{minimize} & \quad c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
\text{subject to} & \quad x_1 + x_2 + x_3 + x_4 = 2 \\
& \quad 2x_1 + 3x_3 + 4x_4 = 2 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

Since columns \( A_1 \) and \( A_2 \) of \( A \) are linearly independent, we choose \( x_1 \) and \( x_2 \) as basic variables. Then

\[
B = \begin{bmatrix} A_1 & A_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}.
\]

Set \( x_3 = x_4 = 0 \), we obtain \( x_1 = 1 \) and \( x_2 = 1 \). The basic solution \( x^* = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \) is nondegenerate. (Thus \( d \) is a feasible direction.)

We construct a feasible direction corresponding to an increase in the nonbasic variable \( x_3 \) by setting \( d_3 = 1 \) and \( d_4 = 0 \). It remains to find \( d_1 \) and \( d_2 \), i.e. \( d_B = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \). Now,

\[
d_B = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -B^{-1}A_3 = -\begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}.
\]

Thus, \( d = \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \).

From \( B^{-1}A_3 = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \), only \( (B^{-1}A_j)_1 > 0 \). Thus we have

\[
\theta^* = \min\{ \frac{(B^{-1}b)_k}{(B^{-1}A_j)_k} \mid (B^{-1}A_j)_k > 0 \} = \min\{ \frac{1}{3/2} \} = 2/3.
\]

At the adjacent basic feasible solution \( x' \) where \( x_3 \) enters as a basic variable, we will have \( x_3 = 2/3 \) and \( x_1 = 0 \), i.e. \( x_1 \) becomes a non basic variable. This adjacent basic feasible solution is

\[
x' = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2/3 \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4/3 \\ 2/3 \\ 0 \end{bmatrix}.
\]
For the objective function $c^T x$, moving from $x^*$ to $x' = x^* + \theta d$, the change on the objective function is

$$c^T x' - c^T x^* = \theta c^T d.$$  

With $d_B = -B^{-1}A_j$, we obtain the rate of change in the objective value with respect to $x_j$ is $c^T d$, since $d_j = 1$.

**Lemma C**

Suppose $d$, with $d_B = -B^{-1}A_j$, is the feasible direction obtained as above. Then

$$c^T d = c_j - c_B^T B^{-1} A_j,$$

where $c_B = (c_{B(1)}, c_{B(2)}, \cdots, c_{B(m)})^T$.

**Proof.** Since $d_j = 1$, we have

$$c^T d = \sum_{i=1}^{n} c_i d_i = \sum_{i=1}^{m} c_{B(i)} d_{B(i)} + c_j = c_B^T d_B + c_j = c_j - c_B^T B^{-1} A_j.$$

**Definition (reduced cost)**

Let $x^*$ be a basic solution, with associated basis matrix $B$. Let $c_B$ be vector of the costs of the basic variables. For each $j, j = 1, 2, \cdots, n$, the reduced cost $\bar{c}_j$ of the variable $x_j$ is defined according to the formula:

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j.$$

**Remark**

Using the reduced costs, we can determine whether moving to an adjacent basic feasible solution improves the objective values. If the $\bar{c}_j < 0$ (respectively $\bar{c}_j > 0$), then moving from $x^*$ to $x' = x^* - \theta B^{-1} A_j$ would decrease (respectively increases) the objective value by $\theta \bar{c}_j$. 

41
Lemma D
For each basic variable \( x_{B(i)}, i = 1, 2, \cdots, m \), the reduced cost \( \bar{c}_{B(i)} = 0 \).

Proof Note that \( B^{-1} \left[ \begin{array}{ccc} A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)} \end{array} \right] = I_m. \)

Thus \( B^{-1}A_{B(i)} = e_i \), the \( i \)th column of \( I_m \).

Hence, \( c^T B^{-1} A_{B(i)} = c_{B(i)} \), the \( i \)th component of \( c_B \). Thus,
\[
\bar{c}_{B(i)} = c_{B(i)} - c^T_B B^{-1} A_{B(i)} = 0.
\]

Example 2.2 Consider the LP problem (refer to Example 2.1),
\[
\begin{align*}
\text{minimize} & \quad x_1 - x_2 + 3x_3 - 4x_4 \\
\text{subject to} & \quad x_1 + x_2 + x_3 + x_4 = 2 \\
& \quad 2x_1 + 3x_3 + 4x_4 = 2 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

(a) For the basic feasible solution \( x^* = (1, 1, 0, 0)^T \), the rate of cost change along the feasible direction (with \( x_3 \) enters as a basic variable) \( d = \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \) is
\[
\bar{c}_3 = 1(-3/2) + (-1)(1/2) + 3(1) + (-4)(0) = 1.
\]
Note that \( c^T d = \bar{c}_3 \). The rate of change along this direction is 1.

(b) For each variable \( x_j \), the reduced cost \( \bar{c}_j = c_j - c^T_B B^{-1} A_j \) are computed as follows:

For \( x_1 \): \( \bar{c}_1 = c_1 - c^T_B B^{-1} A_1 = 0 \).

For \( x_2 \): \( \bar{c}_2 = c_2 - c^T_B B^{-1} A_2 = 0 \).

For \( x_3 \): \( \bar{c}_3 = c_3 - c^T_B B^{-1} A_3 = 1 \).

For \( x_4 \): \( \bar{c}_4 = c_4 - c^T_B B^{-1} A_4 = -7 \).
Theorem (Sufficient conditions for Optimality.)
Consider a basic feasible solution \( x \) associated with a basis matrix \( B \), let \( \bar{c} \) be the corresponding vector of reduced costs.

(a) For a minimization problem, if \( \bar{c} \geq 0 \), then \( x \) is optimal.

(b) For a maximization problem, if \( \bar{c} \leq 0 \), then \( x \) is optimal.

Proof. (a) Assume \( \bar{c} \geq 0 \). Let \( y \) be an arbitrary feasible solution.

(Aim: Show \( c^T y \geq c^T x \).)

Let \( w = y - x \) and note that \( Aw = 0 \). Thus, we have

\[
Bw_B + \sum_{i \in N} A_i w_i = 0,
\]

where \( N \) is the set of indices corresponding to the nonbasic variables.

Since \( B \) is invertible, we obtain

\[
w_B = -\sum_{i \in N} B^{-1} A_i w_i,
\]

and

\[
c^T w = c^T_B w_B + \sum_{i \in N} c_i w_i = \sum_{i \in N} (c_i - c^T_B B^{-1} A_i) w_i = \sum_{i \in N} \bar{c}_i w_i.
\]

For each nonbasic index, \( i \in N \), we must have \( x_i = 0 \) and \( y_i \geq 0 \) so that \( w_i \geq 0 \) and hence \( \bar{c}_i w_i \geq 0 \).

Therefore, \( c^T y - c^T x = c^T (y - x) = c^T w = \sum_{i \in N} \bar{c}_i w_i \geq 0 \). Thus, \( x \) is optimal.

Proposition
Consider a nondegenerate basic feasible solution \( x \) associated with a basis matrix \( B \), let \( \bar{c} \) be the corresponding vector of reduced costs.

For a minimization (respectively maximization) problem, \( x \) is optimal if and only if \( \bar{c} \geq 0 \) (respectively \( \bar{c} \leq 0 \)).

Proof. (We prove the proposition for a minimization problem.)

Suppose \( x \) is nondegenerate feasible solution which is optimal.

(We prove by contradiction that \( \bar{c} \geq 0 \).)

Suppose \( \bar{c}_j < 0 \) for some \( j \). Then \( x_j \) must be nonbasic, by Lemma D.

Since \( x \) is nondegenerate, the direction \( d \) (obtained in Lemma A, where \( d_B = -B^{-1} A_j \)) is a feasible direction, i.e. there is a positive scalar \( \theta \) such that \( x + \theta d \) is a feasible solution.

Since \( \bar{c}_j < 0 \), we have a decrease in the cost at \( x + \theta d \), contradicting \( x \) being optimal. QED.
**Remark** To decide whether a nondegenerate basic feasible solution is optimal, it is enough to check whether all reduced costs of $n - m$ nonbasic variables are nonnegative (respectively nonpositive) if it is a minimization (respectively maximization) problem.

**Warning** In the degenerate case, an optimal basic feasible solution needs not have nonnegative reduced costs.

**Example 2.3** Consider the LP problem

\[
\begin{align*}
\text{minimize} \quad & c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 \\
\text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 2 \\
& 2x_1 + 3x_3 + 4x_4 = 2 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

(a) For the objective

\[
\text{minimize} \quad x_1 - x_2 + 3x_3 - 4x_4,
\]

the basic feasible solution \(x = (1, 1, 0, 0)^T\), with

\[
B = \begin{bmatrix} A_1 & A_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}.
\]

The vector of reduced costs is \(\bar{c} = (0, 0, 1, -7)^T\) (computed in Example 2.2).

Since \(c_4 < 0\) and \(x\) is nondegenerate, \(x\) is not optimal.

(b) For the objective

\[
\text{minimize} \quad x_1 - x_2 + 3x_3 + 4x_4
\]

subject to the same constraints.

The basic feasible solution \(x = (1, 1, 0, 0)^T\), with the same \(B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\) is an optimal solution since \(\bar{c} = (0, 0, 1, 1)^T \geq 0\).

By the next definition, the basis matrix \(B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\) is said to be optimal.
To determine whether a basic solution is optimal, we need to check for feasibility and nonnegativity (or nonpositivity) of the reduced costs. Thus we have the following definition.

**Definition** For a minimization (respectively maximization) problem, a basis matrix $B$ is said to be **optimal** if:

(a) $x_B = B^{-1}b \geq 0$, and

(b) $\bar{c}^T = c^T - c_B^T B^{-1}A \geq 0$ (respectively $\bar{c} \leq 0$).

### 3.3 The Simplex Method.

The simplex method is based fundamentally on the fact that the optimum solution occurs at a corner point of the solution space. It employs an **iterative process** that starts at a **basic feasible solution**, and then attempts to find an adjacent basic feasible solution that will improve the objective value.

This is possible only if an *increase* in a current nonbasic variable can lead to an improvement in the objective value. However, for a current nonbasic variable to become positive, one of the current basic variables must be removed (become nonbasic at zero level) to guarantee that the new solution will include exactly $m$ basic variables. The selected nonbasic variable is called the **entering variable** and the removed basic variable is called the **leaving variable**.

The corner stones of the simplex method are:

1. the optimality conditions, i.e. non-negativity (respectively nonpositivity) of the reduced costs for a minimization (respectively maximization) problem, that allow us to test whether the current basis is optimal;

2. a systematic method for performing basis changes whenever the optimality conditions are violated.
The simplex method is initiated with a starting basic feasible solution (guaranteed for feasible standard form problem), and continues with the following typical iteration.

1. In a typical iteration, we start with a basis consisting of the basic columns \( A_{B(1)}, A_{B(2)}, \ldots, A_{B(m)} \), and an associated basic feasible solution \( \mathbf{x} \).

2. Compute the reduced costs \( \bar{c}_j = c_j - c^T_B B^{-1} A_j \) for all nonbasic variables \( x_j \).

   For a minimization (respectively maximization) problem, if they are all nonnegative (respectively nonpositivity), the current basic feasible solution is optimal, and the algorithm terminates; else, choose some \( j^* \) for which \( \bar{c}_{j^*} < 0 \) (respectively \( \bar{c}_{j^*} > 0 \)).

   The corresponding \( x_{j^*} \) is called the **entering variable**.

3. Compute \( \mathbf{u} = B^{-1} A_{j^*} \).

4. If no component of \( \mathbf{u} \) is positive, we conclude that the problem is unbounded, and the algorithm terminates.

   If some component of \( \mathbf{u} \) is positive, let
   \[
   \theta^* = \min \left\{ \frac{x_{B(i)}}{u_i} \mid u_i > 0 \right\}.
   \]

5. Let \( l \) be such that \( \theta^* = \frac{x_{B(l)}}{u_l} \).

   The corresponding \( x_{B(l)} \) is called the **leaving variable**.

   Form a new basis by replacing \( A_{B(l)} \) with \( A_{j^*} \).

   The entering variable \( x_{j^*} \) assumes value \( \theta^* = \frac{x_{B(l)}}{u_l} \) whereas the other basic variables assume values \( x_{B(i)} - \theta^* u_i \) for \( i \neq l \).
Example 3.1 We shall demonstrate the simplex iteration for the following LP problem in Example 2.3 (a).

\[
\begin{align*}
\text{minimize} \quad & x_1 - x_2 + 3x_3 - 4x_4 \\
\text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 2 \\
& 2x_1 + 3x_3 + 4x_4 = 2 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

1. Start with the basis \( \{A_1, A_2\} \) associated with basic feasible solution \( x = (1, 1, 0, 0)^T \).

2. Compute reduced costs for nonbasic variables, check for optimality and select entering variable if nonoptimal.

   For nonbasic variables \( x_3 \) and \( x_4 \), the respective reduced costs are \( \bar{c}_3 = 1 \) and \( \bar{c}_4 = -7 \).

   Since \( \bar{c}_4 < 0 \), choose \( x_4 \) to be the entering variable.

3. Compute the basic direction correspond to the entering variable.

   The \( x_4 \)-basic direction \( u = B^{-1}A_4 = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \).

4. Check for positive components of \( u \) to select the leaving variable.

   The first component of \( u = u_1 = 2 > 0 \), and \( x_{B(1)} = x_1 = 1 \). Thus

   \[
   \theta^* = \min \left\{ \frac{x_{B(i)}}{u_i} \mid u_i > 0 \right\} = 1/2 = \frac{x_{B(1)}}{u_1}.
   \]

   Therefore, \( x_1 \) is the leaving variable.

5. Determine the new basic feasible solution and basis.

   The new basis is \( A_4 \) and \( A_2 \).

   The entering variable \( x_4 \) assume value \( \theta^* = 1/2 \), the other basic variable \( x_2 \) assumes value \( 1 - (1/2)(-1) = 3/2 \) (from \( x_{B(i)} - \theta^*u_i \), i.e. \( x_2 - (1/2)u_2 \)).
Chapter 4

Implementing the Simplex Method.

To use the simplex method to solve LP, we first convert the LP into the standard form. In this chapter, we shall discuss two simplex algorithms: the simplex tableau implementation and the revised simplex method. In order to perform the simplex method, we need to obtain a basic feasible solution. Thus, we discuss how to obtain a starting basic feasible solution.

Recall from Linear Algebra: the row operation of adding a constant multiple of one row to another row or multiplying a nonzero scalar to one row is called an elementary row operation. Performing an elementary row operation on a matrix $C$ is equivalent to left-multiplying $C$ by an elementary matrix $E$. 
4.1 The Revised Simplex Method.

In the revised simplex method, the matrix $B^{-1}$ is made available at the beginning of each iteration, and the vectors $c^T B^{-1}$ and $B^{-1} A_j$, are computed by a matrix-vector multiplication. To be practical, we need an efficient method for updating the matrix $B^{-1}$ each time there is a change of basis.

Let $B = [A_{B(1)}, A_{B(2)}, \cdots, A_{B(m)}]$ be the basis matrix at the beginning of an iteration, and

$\bar{B} = [A_{B(1)}, \cdots, A_{B(l-1)}, A_j^*, A_{B(l+1)}, \cdots, A_{B(m)}]$ be the basis at the beginning of the next iteration. (Note: they differ by one column.)

Since $B^{-1} B = I$, we have $B^{-1} A_{B(i)} = e_i$, the $i$th unit vector. Thus,

$$B^{-1} \bar{B} = \begin{bmatrix} e_1 & \cdots & e_{l-1} & u & e_{l+1} & \cdots & e_m \end{bmatrix} = \begin{bmatrix} 1 & u_1 & u_2 & \cdots & u_{l-1} & u_{l+1} & \cdots & 1 \\ \end{bmatrix}$$

where $u = B^{-1} A_j = [u_1, \cdots, u_l, \cdots, u_m]^T$ and $u_l > 0$.

In order to change the last matrix to an identity matrix, we apply a sequence of elementary row operations to the last matrix:

(a) For each $i \neq l$, add $l$-th row times $-u_i/u_l$ to the $i$-th row – replacing $u_i$ by 0.

(b) Divide the $l$-th row by $u_l$ – replacing $u_l$ by one.

This sequence of elementary row operations is equivalent to left-multiplying $B^{-1} \bar{B}$ by an invertible matrix $Q$ which is a product of elementary matrices.

Thus, we obtain

$$QB^{-1} \bar{B} = I,$$

which gives:

$$QB^{-1} = \bar{B}^{-1}.$$

This suggests that applying the above sequence of elementary row operations to $B^{-1}$ gives rise to the inverse matrix $\bar{B}^{-1}$ of the next basis matrix $\bar{B}$.
Remark. Like finding an inverse of a matrix or solving a linear system of equations, we may form the \( m \times (m + 1) \) matrix \([B^{-1}u]\). Then the objective is to apply elementary row operations to transform the right column to the unit vector \(e_l\). In doing so, we obtain the inverse of the next basis matrix \(\bar{B}^{-1}\) on the left.

\[
[B^{-1}u] \rightarrow \cdots \rightarrow [\bar{B}^{-1}e_l].
\]

Example 1.1

Let \(B^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & -2 \end{bmatrix}\), and \(u = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix}\). Suppose \(l = 3\).

Find the inverse of the basis matrix \(\bar{B}\) for the next iteration.

Solution. Note that \(l = 3\).

\[
\begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 1 & 2 \\ 4 & -3 & -2 & 2 \end{bmatrix}
\]

\[
R_1 + 4(1/2)R_3 \quad R_2 - 2(1/2)R_3 \quad \quad \rightarrow \begin{bmatrix} 9 & -4 & -1 & 0 \\ 9 & 6 & 3 & 0 \\ 2 & -3 & 1 & 1 \end{bmatrix}
\]

Thus, \(\bar{B}^{-1} = \begin{bmatrix} 9 & -4 & -1 \\ -6 & 6 & 3 \\ 2 & -\frac{3}{2} & -1 \end{bmatrix}\).
The following summarize the steps in a typical iteration of the revised simplex method:

1. In a typical iteration of the revised simplex method, we start with a basis consisting of the basic columns \( A_{B(1)}, A_{B(2)}, \ldots, A_{B(m)} \), an associated basic feasible solution \( \mathbf{x} \), and the inverse \( \mathbf{B}^{-1} \) of the basis matrix.

2. Compute the row vector \( \mathbf{p}^T = \mathbf{c}_B^T \mathbf{B}^{-1} \) and then the reduced costs \( \bar{c}_j = c_j - \mathbf{p}^T \mathbf{A}_j \).
   For a minimization problem (respectively maximization problem), if the reduced costs are all nonnegative (respectively nonpositive), the current basic feasible solution is optimal, and the algorithm terminates; else, choose some \( j^* \) for which \( \bar{c}_{j^*} < 0 \) (respectively \( \bar{c}_{j^*} > 0 \)).

3. Compute \( \mathbf{u} = \mathbf{B}^{-1} \mathbf{A}_{j^*} = (u_1, \ldots, u_l, \ldots, u_m)^T \).

4. If all \( u_i \leq 0 \), we conclude that the problem is unbounded, and the algorithm terminates.
   If some component of \( \mathbf{u} \) is positive, let
   \[
   \theta^* = \min \left\{ \frac{x_{B(i)}}{u_i} \mid u_i > 0 \right\}.
   \]

5. Let \( l \) be such that \( \theta^* = \frac{x_{B(l)}}{u_l} \).
   Form a new basis by replacing \( A_{B(l)} \) with \( A_{j^*} \).
   The entering variable \( x_{j^*} \) assumes value \( \theta^* = \frac{x_{B(l)}}{u_l} \) whereas the other basic variables assume values \( x_{B(i)} - \theta^* u_i \) for \( i \neq l \).

6. Form the \( m \times (m + 1) \) matrix \( [\mathbf{B}^{-1}|\mathbf{u}] \). Add to each one of its rows a multiple of the \( l \)th row to make the last column equal to the unit vector \( \mathbf{e}_l \). The first \( m \) columns of the result is the matrix \( \mathbf{B}^{-1} \).
**Example 1.2** We demonstrate an iteration of the revised simplex for the following LP problem.

\[
\begin{align*}
\text{minimize} & \quad x_1 - x_2 + 3x_3 - 4x_4 \\
\text{subject to} & \quad x_1 + x_2 + x_3 + x_4 = 2 \\
& \quad 2x_1 + 3x_3 + 4x_4 = 2 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

1. Start with the basis matrix \( B = [A_1, A_2] = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \), associated with
basic feasible solution \( x = (1, 1, 0, 0)^T \), and \( B^{-1} = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix} \).

2. Compute \( p^T = c^T_B B^{-1} \) and reduced costs \( \bar{c}_j = c_j - p^T A_j \) for non basic variables.

\[
p^T = c^T_B B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1 \end{bmatrix}.
\]

Nonbasic variables are \( x_3 \) and \( x_4 \), with the respective reduced costs

\[
\bar{c}_3 = 3 - \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} = 1 \text{ and } \bar{c}_4 = -4 - \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1 \end{bmatrix} = -7.
\]

Since \( \bar{c}_4 < 0 \), choose \( x_4 \) to be the entering variable.

3. Compute \( u = B^{-1} A_4 \) correspond to the entering variable.

\[
u = B^{-1} A_4 = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/1 \end{bmatrix}.
\]

4. Check for positive components of \( u \) to select the leaving variable.

The first component of \( u = u_1 = 2 > 0 \), and \( x_{B(1)} = x_1 = 1 \). Thus

\[
\theta^* = \min \left\{ \frac{x_{B(i)}}{u_i} \mid u_i > 0 \right\} = 2/1 = \frac{x_{B(1)}}{u_1}.
\]

Therefore, \( x_1 \) is the leaving variable.

5. Form a new basis replacing \( A_1 \) by \( A_4 \), i.e. \( \bar{B} = [A_4, A_2] \).

The entering variable \( x_4 \) assumes value \( \theta^* = 1/2 \), the other basic variable \( x_2 \) assumes value \( 1 - (1/2)(-1) = 3/2 \) (from \( x_{B(i)} - \theta^* u_i \), i.e. \( x_2 - (1/2)u_2 \)).

6. Find the inverse \( \bar{B}^{-1} \) of the next basis matrix \( \bar{B} \).

\[
[B^{-1}|u] = \begin{bmatrix} 0 & 1/2 & 2 \\ 1 & -1/2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1/4 & 1 \\ 1 & -1/4 & 0 \end{bmatrix} = [\bar{B}^{-1}|e_1].
\]

52
4.2 Simplex Tableau Implementation.

For a selected starting basic feasible solution, we form the simplex tableau and carry out the remaining iterative steps to obtain the optimal solution or conclude that the optimal cost is infinite.

General Simplex Tableau.

Consider the standard form of an LP,

\[
\begin{align*}
\text{Minimize} & \quad c^T x \quad \text{(or Maximize} \quad cx) \\
\text{Subject to} & \quad Ax = b \\
& \quad x \geq 0.
\end{align*}
\]

For a basic feasible solution with basis \( B \), we form the following simplex tableau for the implementation of the simplex algorithm:

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x )</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{c} )</td>
<td>( c^T - c^T_B B^{-1} A )</td>
<td>( -c^T_B B^{-1} b )</td>
</tr>
<tr>
<td>( x_B )</td>
<td>( B^{-1} A )</td>
<td>( B^{-1} b )</td>
</tr>
</tbody>
</table>

In detail, ( \( \bar{c}_j = c_j - c^T_B B^{-1} A_j \) )

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( \cdots )</th>
<th>( x_n )</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{c} )</td>
<td>( \bar{c}_1 )</td>
<td>( \cdots )</td>
<td>( \bar{c}_n )</td>
<td>( -c^T_B x_B )</td>
</tr>
<tr>
<td>( x_{B(1)} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \cdots )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_{B(i)} )</td>
<td>( B^{-1} A_1 )</td>
<td>( \cdots )</td>
<td>( B^{-1} A_n )</td>
<td>( B^{-1} b )</td>
</tr>
<tr>
<td>( \cdots )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_{B(m)} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes

1. For each basic variable, \( x_{B(i)} \), we have \( B^{-1} A_{B(i)} = e_i \).
2. If \( A_i = e_i \), then \( B^{-1} A_i = \) the \( i \)th column of \( B^{-1} \).
Example 2.1 Consider the following LP problem,

$$\begin{align*}
\text{minimize} & \quad x_1 - x_2 + 3x_3 - 4x_4 \\
\text{subject to} & \quad x_1 + x_2 + x_3 + x_4 = 2 \\
& \quad 2x_1 + 3x_3 + 4x_4 = 2 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}$$

Write down the simplex tableau associated with the basis matrix $B = [A_1, A_2]$.

Solution. To obtain the simplex tableau, we need to compute reduced costs $\bar{c}$, $B^{-1}A$, $x_B = B^{-1}b$, and $-c_B^TB^{-1}b$.

Note that

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \quad \text{and} \quad B^{-1} = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix}.$$

Thus,

$$x_B = B^{-1}b = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

$$B^{-1}A = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3/2 & 2 \\ 0 & 1 & -1/2 & -1 \end{bmatrix};$$

$$\bar{c}^T = c^T - c_B^TB^{-1}A = (1, -1, 3, -4) - (1, -1) \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 3 & 4 \end{bmatrix} = (0, 0, 1, -7);$$

and

$$-c_B^TB^{-1}b = -c_B^Tx_B = -(1, -1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0.$$

Therefore the corresponding simplex tableau is:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-7</td>
<td>0</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>3/2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

54
Example 2.2 Consider the following LP problem

Minimize \(-3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4\)
Subject to \(x_1 + 2x_2 + s_1 = 6\)
\(2x_1 + x_2 + s_2 = 8\)
\(-x_1 + x_2 + s_3 = 1\)
\(x_2 = 2\)
\(x_1, x_2, s_1, s_2, s_3, s_4 \geq 0\)

Determine the simplex tableau corresponds to

(a) \(B = I\) associated with basic variables: \(s_1, s_2, s_3, s_4\).

(b) \(B\) associated with basic variables: \(x_1, x_2, s_3, s_4\).

Solution

(a) Note that: \(c_B = (0, 0, 0, 0)^T\) and thus \(\bar{c}^T = c^T - c_B^T B^{-1} A = (-3, -2, 0, 0, 0, 0)\).
Since \(B = I\), we have \(B^{-1} = I\). Thus, \(B^{-1} A = A\), \(x_B = B^{-1} b = b\), and hence \(-c_B^T B^{-1} b = 0\).

The simplex tableau corresponds to \(B = I\) associated with basic variables: \(s_1, s_2, s_3, s_4\) is

<table>
<thead>
<tr>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
<th>(s_4)</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>-3</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(s_1)</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>(s_2)</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>(s_3)</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(s_4)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

(b) Note that

1. \(B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\) and \(B^{-1} = \begin{bmatrix} -\frac{1}{3} & 2 & 0 & 0 \\ -\frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & -\frac{1}{3} & 0 & 1 \end{bmatrix}\).

2. \(c_B = (-3, -2, 0, 0)^T\), and \(\bar{c}^T = c^T - c_B^T B^{-1} A = (0, 0, \frac{1}{3}, \frac{4}{3}, 0, 0)\).

3. \(B^{-1} A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & -\frac{2}{3} & -\frac{1}{3} & 0 & 1 \end{bmatrix}\). and \(x_B = B^{-1} b = \begin{bmatrix} 10 & \frac{3}{3} \\ \frac{3}{3} & \frac{3}{3} \\ \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}\), hence

\(-c_B^T B^{-1} b = 12\frac{2}{3}\).

Thus, the simplex tableau corresponds to \(B\) associated with basic variables: \(x_1, x_2, s_3, s_4\), is

<table>
<thead>
<tr>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
<th>(s_4)</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{3})</td>
<td>(\frac{3}{3})</td>
<td>0</td>
<td>0</td>
<td>(12\frac{2}{3})</td>
</tr>
<tr>
<td>(x_1)</td>
<td>1</td>
<td>0</td>
<td>-(\frac{1}{3})</td>
<td>(\frac{3}{3})</td>
<td>0</td>
<td>0</td>
<td>(10)</td>
</tr>
<tr>
<td>(x_2)</td>
<td>0</td>
<td>1</td>
<td>(\frac{1}{3})</td>
<td>-(\frac{3}{3})</td>
<td>0</td>
<td>0</td>
<td>(\frac{3}{3})</td>
</tr>
<tr>
<td>(s_3)</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>(s_4)</td>
<td>0</td>
<td>0</td>
<td>-(\frac{2}{3})</td>
<td>(\frac{1}{3})</td>
<td>0</td>
<td>1</td>
<td>(\frac{2}{3})</td>
</tr>
</tbody>
</table>
The Simplex Tableau Algorithm

The steps of a typical simplex algorithm are summarized as follows:

1. A typical iteration starts with the tableau associated with a basis matrix $B$ and the corresponding basic feasible solution $x$.

2. Examine the reduced costs in the $\bar{c}$-row of the tableau.
   
   For a minimization (respectively maximization) problem, if they are all nonnegative (respectively nonpositive), the current basic feasible solution is optimal, and the algorithm terminates;
   
   else, choose the entering variable to be the nonbasic variable $x_j^*$ for which $\bar{c}_j < 0$ (respectively $\bar{c}_j > 0$).

3. Consider the $j^*$-column, known as pivot column, (which is $u = B^{-1}A_{j^*}$) of the tableau.
   
   If every component of $u$ is nonpositive, i.e. $u_i \leq 0 \forall i$, then the problem is unbounded, and the algorithm terminates.
   
   Else, for each $i$ for which $u_i > 0$, compute the ratio $\frac{x_{B(i)}}{u_i}$.
   
   Choose the leaving variable to be $x_{B(l)}$ corresponding to the smallest ratio.
   
   (Note: $A_{B(l)}$ leaves the basis and $A_{j^*}$ enters the basis.)

4. Add to each row of the tableau a constant multiple of the $l$th row (known as the pivot row) so that $u_l$ (known as the pivot element) becomes one and all other entries of the pivot column become zero.
Example 2.3 Consider the following LP problem

\[
\begin{align*}
\text{Minimize} & \quad -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 \\
\text{Subject to} & \quad x_1 + 2x_2 + s_1 \quad = 6 \\
& \quad 2x_1 + x_2 + s_2 \quad = 8 \\
& \quad -x_1 + x_2 + s_3 \quad = 1 \\
& \quad x_2 \quad + s_4 \quad = 2 \\
& \quad x_1, x_2, s_1, s_2, s_3, s_4 \geq 0
\end{align*}
\]

We shall demonstrate an iteration of the simplex iteration on the following simplex tableau, associated with \(B = I\), where \(s_1, s_2, s_3, s_4\) are basic variables.

<table>
<thead>
<tr>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
<th>(s_4)</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s_1)</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>(s_2)</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>(s_3)</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(s_4)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

**Step 1.** This iteration starts with \(B = I\), and basic feasible solution \((x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 8, 1, 2)\).

**Step 2.** Check for optimality: Is there any negative value in the \(\bar{c}\)-row? (Note that this is a minimization LP.)

Among the reduced costs of nonbasic variables, \(x_1\) and \(x_2\), in the \(\bar{c}\)-row, that of \(x_1\) is negative. Thus, \(x_1\) is selected as the entering variable and \(x_1\)-column is the pivot column. \((x_2\) can also be chosen as the entering variable.)

**Step 3.** Select a leaving variable from the current basic variables with the minimum ratio.

Comparing ratios (with positive denominators),

<table>
<thead>
<tr>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
<th>(s_4)</th>
<th>(\text{Soln})</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{c})</td>
<td>-3</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Pivot row →

| \(s_1\) | 1 | 2 | 1 | 0 | 0 | 0 | 6 | 6 |
| \(s_2\) | 2 | 1 | 0 | 1 | 0 | 0 | 8 | 4 |
| \(s_3\) | -1 | 1 | 0 | 0 | 1 | 0 | 1 | 
| \(s_4\) | 0 | 1 | 0 | 0 | 0 | 1 | 2 | 

Since \(s_2\)-row is associated with the smallest ratio, \(s_2\) will be the leaving variable and \(s_2\)-row is a pivot row.
Step 4. Determine the new basic solution (the next simplex tableau), via row operations, to make the pivot column a unit column. This makes the entering variable a basic variable and the leaving variable a nonbasic one.

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>0</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{3}{2}$</td>
<td>0</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>$x_1$ enters</td>
<td>$s_1$</td>
<td>0</td>
<td>$\frac{3}{2}$</td>
<td>1</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$x_2$</td>
<td>1</td>
<td>$\frac{3}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$ leaves</td>
<td>$s_4$</td>
<td>0</td>
<td>$\frac{3}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The new basic feasible solution is

$$(x_1, x_2, s_1, s_2, s_3, s_4) = (4, 0, 2, 0, 5, 2)$$

with cost $-12$ decreased from 0.

Justification for performing row operations.

From $B$ to $B_1$, where $x_{j*}$ enters and $x_{B(l)}$ leaves.

Let $Q$ be the product of elementary matrices applied to $B_{-1}$ to obtain $B_{1-1}$.

Then $B_{1-1} = QB^{-1}$

1. $B_{1-1}A = QB^{-1}A$ and

   $$x_{B_1} = B_{1-1}b = QB^{-1}b = Qx_B.$$ 

   Thus, by applying row operations to the block $B_{1-1}A$ and the column $B_{1-1}b$ in simplex tableau associated with $B$, we obtain the block $B_{1-1}A$ and the column $B_{1-1}b$ in simplex tableau associated with $B_1$.

2. Change in objective value

   $$\bar{c}_j^* \theta^* = \bar{c}_{j*} \frac{(B^{-1}b)_l}{(B^{-1}A_{j*})_l} = \left( \frac{\bar{c}_{j*}}{(B^{-1}A_{j*})_l} \right) (B^{-1}b)_l.$$ 

3. Denote new reduced cost of $x_k$ by $\bar{c}_k'$. Then

   $$\bar{c}_k' - \bar{c}_k = -\left(c_{B_1}^T B_{1-1} - c_B^T B^{-1}\right) A_k = -\left(c_{B_1}^T Q - c_B^T \right) B_{1-1} A_k.$$ 

Check: All components, except $l$-component, of $c_{B_1}^T Q - c_B^T$ are zero.

The $l$-component of $c_{B_1}^T Q - c_B^T$ is

$$\frac{\bar{c}_{j*}}{(B^{-1}A_{j*})_l}.$$ 

Thus,

$$\bar{c}_k' - \bar{c}_k = -\frac{\bar{c}_{j*}}{(B^{-1}A_{j*})_l} \times (B^{-1}A_k)_l.$$
4.3 Starting the Simplex Algorithms.

The initial step to implement the simplex method is to have a basic feasible solution. Once we have a basic feasible solution, we can perform the simplex iterations to solve the given problem.

In this section, we shall look at how to obtain a basic feasible solution. We shall consider the minimization (or maximization) problem with nonnegative right-hand side ($b \geq 0$). (By possibly multiplying some of the constraints by $-1$, we can assume, that $b \geq 0$.)

If all functional constraints are of type $\leq$, then a ready starting basic feasible solution is available. However, if some functional constraints are of type $\geq$ or $=$, then we have to add artificial variables in order to obtain a basic feasible solution to the modified LP.

(i) Functional constraints of type $\leq$.

Consider the minimization (or maximization) problem where all functional constraints are of $\leq$ type, with nonnegative right-hand side ($b \geq 0$). Assume $A$ is $m \times n$.

Minimize $c^T x$
Subject to $Ax \leq b$
$x \geq 0$.

For such model, each constraint is associated with a slack variable. Thus, the number of slack variables equals the number of functional constraints. The corresponding standard form LP is:

Minimize $c^T x + 0S$
Subject to $Ax + S = b$, i.e. $[A, I] \begin{bmatrix} x \\ S \end{bmatrix} = b$,
$x, S \geq 0$.

The matrix $[A, I]$ is $m \times (m + n)$ and there are $m + n$ decision variables. Thus, a basic feasible solution $\begin{bmatrix} x \\ S \end{bmatrix}$ must satisfy $[A, I] \begin{bmatrix} x \\ S \end{bmatrix} = b$ and there are $n$ nonbasic variables.

Choosing $x$ to be nonbasic variables, and $S = b$ to be basic variables provides a starting basic feasible solution to carry out the simplex iteration.
The starting simplex tableau associated with this basis is:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{c} = c$</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$s_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\cdot$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s_i$</td>
<td>$A_1$</td>
<td>$\cdots$</td>
<td>$A_n$</td>
<td>$I$</td>
<td></td>
<td></td>
<td>$b$</td>
</tr>
<tr>
<td>$\cdot$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s_m$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Once we obtain a basic feasible solution to a given linear programming problem, we may apply the simplex algorithm to solve the problem.

**Example 3.1** The standard LP form of the following:

Minimize $-3x_1 - 2x_2$

Subject to $x_1 + 2x_2 \leq 6$
$2x_1 + x_2 \leq 8$
$-x_1 + x_2 \leq 1$
$x_2 \leq 2$

$x_1, x_2 \geq 0$

is

Minimize $-3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

Subject to $x_1 + 2x_2 + s_1 = 6$
$2x_1 + x_2 + s_2 = 8$
$-x_1 + x_2 + s_3 = 1$
$x_2 + s_4 = 2$

$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$

**Step 0.** A readily available starting basic feasible solution is:

basic variables: slack variables $s_1, s_2, s_3, s_4,$

nonbasic variables: $x_1, x_2 = 0,$

associated basis matrix $B = I.$

Starting tableau:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{c} = -3$</td>
<td>$-2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$s_1$</td>
<td>$1$</td>
<td>$2$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$6$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$2$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$8$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

**Step 1.** Check for optimality. Is there any negative value in the $\bar{c}$-row?
The reduced costs of both nonbasic variables, $x_1$ and $x_2$, are negative. We choose $x_1$ as an entering variable. Column $x_1$ is the pivot column.

**Step 2.** Select a leaving variable from the current basic variables to be a nonbasic variable when the entering variable becomes basic.

Comparing ratios (with positive denominators),

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>Soln ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$-3$</td>
<td>$-2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$s_1$</td>
<td>$1$</td>
<td>$2$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$6$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$2$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$8$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

$s_2$-row is associated with the smallest ratio. Thus $s_2$ will be the leaving variable and $s_2$-row is the pivot row.

**Step 3.** Determine the new basic solution (the next simplex tableau), via row operations, by making the entering variable basic and the leaving variable nonbasic. Go to Step 1.

New Tableau

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$0$</td>
<td>$-\frac{1}{2}$</td>
<td>$0$</td>
<td>$\frac{3}{2}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$12$</td>
</tr>
<tr>
<td>$s_1$</td>
<td>$0$</td>
<td>$\frac{4}{3}$</td>
<td>$1$</td>
<td>$-\frac{1}{2}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$2$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$1$</td>
<td>$\frac{1}{2}$</td>
<td>$0$</td>
<td>$\frac{3}{2}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$4$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$0$</td>
<td>$\frac{3}{2}$</td>
<td>$0$</td>
<td>$\frac{1}{2}$</td>
<td>$1$</td>
<td>$0$</td>
<td>$5$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

Thus, the new basic feasible solution is $(x_1, x_2, s_1, s_2, s_3, s_4) = (4, 0, 2, 0, 5, 2)$ with cost $-12$ decreased from 0.

**Optimum**

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{4}{3}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$12$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$\frac{2}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$-\frac{1}{3}$</td>
<td>$\frac{2}{3}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$3$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-\frac{2}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$0$</td>
<td>$1$</td>
<td>$\frac{2}{3}$</td>
</tr>
</tbody>
</table>

The solution yields $(x_1, x_2, s_1, s_2, s_3, s_4) = (\frac{10}{3}, \frac{4}{3}, 0, 0, 3, 2/3)$ with cost $-12\frac{2}{3}$ decreased from $-12$.

The last tableau is optimal because none of the nonbasic variables (i.e. $s_1$ & $s_2$) has a negative reduced cost in the $\bar{c}$-row.

Graphically, the simplex algorithm starts at the origin $A$ (starting solution) and moves to an adjacent corner point at which the objective
value could be improved. At $B$ ($x_1 = 4, x_2 = 0$), the objective value will be decreased. Thus $B$ is a possible choice. The process is repeated to see if there is another corner point that can improve the value of the objective function. Eventually, the algorithm will stop at $C$ (i.e. $x_1 = \frac{10}{3}, x_2 = \frac{4}{3}$) (the optimum). Hence it takes 3 iterations ($A, B$ and $C$) to reach the optimum.

Putting all tableaus together:

<table>
<thead>
<tr>
<th></th>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>Soln ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>$c$</td>
<td>$-3$</td>
<td>$-2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$s_1$</td>
<td>$1$</td>
<td>$2$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$6$</td>
</tr>
<tr>
<td></td>
<td>$s_2$</td>
<td>$2$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$8$</td>
</tr>
<tr>
<td></td>
<td>$s_3$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td></td>
<td>$s_4$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1)</td>
<td>$c$</td>
<td>$0$</td>
<td>$-\frac{2}{3}$</td>
<td>$0$</td>
<td>$\frac{2}{3}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$12$</td>
</tr>
<tr>
<td></td>
<td>$s_1$</td>
<td>$0$</td>
<td>$\frac{1}{3}$</td>
<td>$1$</td>
<td>$-\frac{1}{3}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$2$</td>
</tr>
<tr>
<td></td>
<td>$s_2$</td>
<td>$1$</td>
<td>$\frac{1}{3}$</td>
<td>$0$</td>
<td>$\frac{1}{3}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$4$</td>
</tr>
<tr>
<td></td>
<td>$s_3$</td>
<td>$0$</td>
<td>$\frac{1}{3}$</td>
<td>$0$</td>
<td>$\frac{1}{3}$</td>
<td>$1$</td>
<td>$0$</td>
<td>$5$</td>
</tr>
<tr>
<td></td>
<td>$s_4$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2)</td>
<td>$c$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{1}{3}$</td>
<td>$-\frac{2}{3}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$12$</td>
</tr>
<tr>
<td></td>
<td>$x_2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$-\frac{2}{3}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td></td>
<td>$x_1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$-\frac{1}{3}$</td>
<td>$\frac{2}{3}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{10}{3}$</td>
</tr>
<tr>
<td></td>
<td>$s_3$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$3$</td>
</tr>
<tr>
<td></td>
<td>$s_4$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-\frac{2}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$0$</td>
<td>$1$</td>
<td>$\frac{2}{3}$</td>
</tr>
</tbody>
</table>
Example 3.2

Maximize \[-5x_1 + 4x_2 - 6x_3 - 8x_4\]
Subject to
\[
\begin{align*}
    x_1 &+ 7x_2 + 3x_3 + 7x_4 &\leq 46 \\
    3x_1 - 2x_2 + x_3 + 2x_4 &\leq 8 \\
    2x_1 + 3x_2 - x_3 + x_4 &\leq 10 \\
    x_1, x_2, x_3, x_4 &\geq 0
\end{align*}
\]

The associated standard form LP:

Maximize \[-5x_1 + 4x_2 - 6x_3 - 8x_4 + 0s_1 + 0s_2 + 0s_3\]
Subject to
\[
\begin{align*}
    x_1 &+ 7x_2 + 3x_3 + 7x_4 + s_1 &\leq 46 \\
    x_1 - 2x_2 + x_3 + 2x_4 + s_2 &\leq 8 \\
    2x_1 + 3x_2 - x_3 + x_4 + s_3 &\leq 10 \\
    x_1, x_2, x_3, x_4, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

Thus, the implementation of simplex method via simplex tableaus:

<table>
<thead>
<tr>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{c})</td>
<td>(-5)</td>
<td>(4)</td>
<td>(-6)</td>
<td>(-8)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
</tr>
<tr>
<td>(s_1)</td>
<td>(1)</td>
<td>(7)</td>
<td>(3)</td>
<td>(7)</td>
<td>(1)</td>
<td>(0)</td>
<td>(0)</td>
<td>(46)</td>
</tr>
<tr>
<td>(s_2)</td>
<td>(3)</td>
<td>(-2)</td>
<td>(1)</td>
<td>(2)</td>
<td>(0)</td>
<td>(1)</td>
<td>(0)</td>
<td>(8)</td>
</tr>
<tr>
<td>(s_3)</td>
<td>(2)</td>
<td>(3)</td>
<td>(-1)</td>
<td>(1)</td>
<td>(0)</td>
<td>(0)</td>
<td>(1)</td>
<td>(10)</td>
</tr>
</tbody>
</table>

\(x_2\) enters, \(s_3\) leaves

Thus the optimal solution is \((x_1, x_2, x_3, x_4) = (0, \frac{10}{3}, 0, 0)\) with optimal cost \(\frac{40}{3}\).
Example 3.3 Solve the following problem by the revised simplex method:

Minimize \[ 2x_1 + x_2 + 0x_3 + 3x_4 + 4x_5 \]
Subject to \[ \begin{align*} 3x_1 &+ x_2 &+ 4x_4 &= 3 \\ 4x_1 &+ 3x_2 &- x_3 &+ x_5 &= 6 \\ x_1 &+ 2x_2 & &+ x_6 &= 3 \end{align*} \]
\[ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \]

Solution Note that the columns of \( x_4, x_5 \) and \( x_6 \) form an identity matrix. This gives a readily available basic feasible solution with basic variables \( x_4 = 3, x_5 = 6 \) and \( x_6 = 3 \).

First Iteration (\( B_1 \))

Step 1 \( B_1 = I \) and thus \( B_1^{-1} = I \), and \( x_{B_1} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} \).

Step 2 Compute \( p^T = c^T_{B_1} B_1^{-1} \) and \( \bar{c}_j \) for non-basic variables:

Basic variables are \( x_4, x_5, x_6 \), and \( c^T_{B_1} = (3, 4, 0) \) from the coefficients of \( x_4, x_5, x_6 \) in the objective function.

(a) \( p^T = c^T_{B_1} B_1^{-1} = (3, 4, 0) \).

(b) For nonbasic variables \( x_1, x_2, x_3 \), we compute the reduced costs.

\[ \bar{c}_1, \bar{c}_2, \bar{c}_3 = (c_1, c_2, c_3) - p^T(A_1, A_2, A_3) \]
\[ = (2, 1, 0) - (3, 4, 0) \begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & -1 \\ 1 & 2 & 0 \end{pmatrix} = (-23, -14, 4) \]

Thus, we may choose \( A_1 \) as the entering column (i.e. \( x_1 \) enters).

Step 3 Compute \( u = B_1^{-1} A_1 = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \).

Step 4 Compute the minimum ratio \( \theta^* \) to determine the leaving variable given that \( A_1 \) enters the basis.

(a) \( x_{B_1} = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = B_1^{-1} b = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} \).

(b) \( \theta^* = \min \{3/3, 6/4, 3/1\} = 3/3 \) (from \( x_4 \)).

Step 5 Therefore \( x_4 \) is the leaving variable, i.e. \( A_4 \) is the leaving column.

\[ x_4 = \theta^* = 1, \ x_5 = 6 - \theta^*(4) = 2 \quad \text{and} \quad x_6 = 3 - \theta^*(1) = 2. \]

The new basis matrix \( B_2 = [A_1, A_5, A_6] \), with associated basic solution \( (1, 0, 0, 0, 2, 2) \).
Step 6 Determine the next basis inverse.

\[
[B_1^{-1}]u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} = [B_2^{-1}]e_1
\]

**Second Iteration (B_2)**

Step 1 \( B_2^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} \) with \( x_{B_2} = \begin{pmatrix} x_1 \\ x_5 \\ x_6 \end{pmatrix} \), and \( c_{B_2} = (2, 4, 0)^T \).

Step 2 Computation of \( p^T = c_{B_2}B_2^{-1} \) and \( \bar{c}_j \) for non-basic variables:

(a) \( p^T = (c_1, c_5, c_6) \begin{bmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} = (2, 4, 0) \begin{bmatrix} 1/3 & 0 \\ -4/3 & 1 \\ -1/3 & 0 \end{bmatrix} = (-14/3, 4, 0). \)

(b) For nonbasic variables \( x_2, x_3, x_4 \), we compute the reduced costs.

\[
(\bar{c}_2, \bar{c}_3, \bar{c}_4) = (c_2, c_3, c_4) - p^T(A_2, A_3, A_4)
\]

\[
= (1, 0, 3) - (-14/3, 4, 0) \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = (-19/3, 4, 23/3)
\]

Thus, we choose \( A_2 \) as the entering column (i.e. \( x_2 \) enters).

Step 3 Compute \( u = B_2^{-1}A_2 = \begin{bmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 5/3 \end{pmatrix} \).

Step 4 To determine the leaving variable given that \( A_2 \) enters the basis, we compute the minimum ratio \( \theta^* \).

(a) \( x_{B_2} = \begin{pmatrix} x_1 \\ x_5 \\ x_6 \end{pmatrix} = B_2^{-1}b = \begin{pmatrix} 1/3 \\ -4/3 \\ -1/3 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

(b) \( \theta^* = \min \{1/(1/3), 2/(5/3), 2/(5/3)\} = 6/5 \) (from \( x_5 \) and \( x_6 \)).

Step 5 Therefore \( x_5 \) is the leaving variable, i.e. \( A_5 \) is the leaving column. (Breaking tie arbitrarily.)

\[
x_2 = \theta^* = 6/5, x_1 = 1 - \theta^*(1/3) = 3/5, \text{ and } x_6 = 2 - \theta^*(5/3) = 0.
\]

The new basis matrix \( B_3 = [A_1, A_2, A_6] \), with associated basic solution \((3/5, 6/5, 0, 0, 0, 0, 0)\).

Step 6 Determine the next basis inverse.

\[
[B_2^{-1}]u = \begin{bmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3/5 & -1/5 & 0 & 0 \\ -4/5 & 3/5 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}
\]
Third Iteration \((B_3)\)

Step 1 \(B_3^{-1} = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix}\) with \(x_{B_2} = \begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix}\), and \(C_{B_3} = (2, 1, 0)^T\).

Step 2 Compute of \(p^T = c_{B_3}^T B_3^{-1}\) and \(\bar{c}_j\) for non-basic variables:

(a) \(p^T = (2/5, 1/5, 0)\).

(b) For nonbasic variables \(x_3, x_4, x_5\), we compute the reduced costs.

\[
(\bar{c}_3, \bar{c}_4, \bar{c}_5) = (c_3, c_4, c_5) - p^T (A_3, A_4, A_5)
\]

\[
= (0, 3, 4) - (2/5, 1/5, 0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
= (1/5, 13/5, 19/5)
\]

Thus the current basic feasible solution is optimal since these reduced costs are non-negative.

Therefore the optimal solution is

\[
x_{B_3} = \begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix}
\]

and the optimal cost is \(c_{B_3} x_{B_3} = 12/5\).

Alternatively, by simplex tableau implementation:

<table>
<thead>
<tr>
<th>(B_1)</th>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
<th>Soln</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{c})</td>
<td>-23</td>
<td>-14</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-33</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pivot row</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_4)</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(x_5)</td>
<td>4</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>6</td>
<td>3/2</td>
<td></td>
</tr>
<tr>
<td>(x_6)</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>(B_2)</td>
<td>(\bar{c})</td>
<td>0</td>
<td>-19/3</td>
<td>4</td>
<td>23/3</td>
<td>0</td>
<td>0</td>
<td>-10</td>
<td></td>
</tr>
<tr>
<td>Pivot row</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_1)</td>
<td>1</td>
<td>1/3</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>(x_5)</td>
<td>0</td>
<td>5/3</td>
<td>-1</td>
<td>-4/3</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>6/5</td>
<td></td>
</tr>
<tr>
<td>(x_6)</td>
<td>0</td>
<td>5/3</td>
<td>0</td>
<td>-1/3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>6/5</td>
<td></td>
</tr>
<tr>
<td>(B_3)</td>
<td>(\bar{c})</td>
<td>0</td>
<td>0</td>
<td>1/5</td>
<td>13/5</td>
<td>19/5</td>
<td>0</td>
<td>-12/5</td>
<td></td>
</tr>
<tr>
<td>Pivot row</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_1)</td>
<td>1</td>
<td>0</td>
<td>1/5</td>
<td>3/5</td>
<td>-1/5</td>
<td>0</td>
<td>3/5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_2)</td>
<td>0</td>
<td>1</td>
<td>-3/5</td>
<td>-4/5</td>
<td>3/5</td>
<td>0</td>
<td>6/5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_6)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>6/5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
(ii) Some functional constraints of type \((=)\) or \((\geq)\).

If some functional constraint is an equation \((=)\) or of the type \((\geq)\), we may not have a ready starting basic feasible solution. To obtain a starting basic feasible solution, we introduce **artificial variables** that assume the role of slacks.

As artificial variables have no physical meaning, they must be forced to zero when the optimum is reached, otherwise the resulting solution is infeasible. Two (closely related) methods based on the idea of driving out the artificial variables are devised for this purpose, namely:

(a) The Two-Phase Method.

(b) The Big-\(M\) Method (or \(M\) simplex method).

**How to add artificial variables?**

1. For each constraint of type \(" a_T^i x = b_i \)\), we add an artificial variable \(y_i \geq 0\) to have the modified constraint \(a_T^i x + y_i = b_i\).

2. For each constraint of type \(" a_T^i x \geq b_i \)\), after adding a surplus variable \(s_i \geq 0\), we add an artificial variable \(y_i \geq 0\) to have the modified constraint \(a_T^i x - s_i + y_i = b_i\).

**Example 3.4** Consider the LP problem

\[
\begin{align*}
\text{Minimize} & \quad 4x_1 + x_2 \\
\text{Subject to} & \quad 3x_1 + x_2 = 3 \\
& \quad -4x_1 - 3x_2 \leq -6 \\
& \quad x_1 + 2x_2 \leq 4 \\
& \quad x_1, \ x_2 \geq 0.
\end{align*}
\]

Add artificial variables where necessary and write down the modified constraints.

**Solution**

1. Add artificial variable \(y_1 \geq 0\) to the first constraint:

\[3x_1 + x_2 + y_1 = 3.\]

2. Multiply the second constraint by \((-1)\) to obtain nonnegative \(b\): \(4x_1 + 3x_2 \geq 6\). Add a surplus variable \(s_1 \geq 0\) and an artificial variable \(y_2 \geq 0\):

\[4x_1 + 3x_2 - s_1 + y_2 = 6.\]
(a) The Two-Phase Method.

Introduce artificial variables $y_i$, if necessary, and form the auxiliary LP problem, with the following modified objective and constraints:

The Auxiliary LP problem

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{k} y_i = y_1 + y_2 + \cdots + y_k \\
\text{subject to} & \quad \text{original constraints with slack variables,} \\
& \quad \text{modified constraints,} \\
& \quad x \geq 0, \\
& \quad s_i \geq 0 \text{ for slack and surplus variables } s_i, \\
& \quad y_i \geq 0 \text{ for artificial variable } y_i.
\end{align*}
\]

A ready starting basic feasible solution for the auxiliary LP problem is obtained by choosing basic variables to be artificial variables $y_i$ and slack variables $s_i$ (nonbasic variables are $x$ and surplus variables $s_i$, all assuming zero values), and the associated basis matrix $B = I$.

Example 3.5 For the LP problem:

\[
\begin{align*}
\text{Minimize} & \quad 4x_1 + x_2 \\
\text{Subject to} & \quad 3x_1 + x_2 = 3 \\
& \quad -4x_1 - 3x_2 \leq -6 \\
& \quad x_1 + 2x_2 \leq 4 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

Write down the auxiliary LP problem and a basic feasible solution for the auxiliary LP problem.

**Solution**

Refer to the Example 3.4, the auxiliary LP problem

\[
\begin{align*}
\text{Minimize} & \quad y_1 + y_2 \\
\text{Subject to} & \quad 3x_1 + x_2 + y_1 = 3 \\
& \quad 4x_1 + 3x_2 - s_1 + y_2 = 6. \\
& \quad x_1, x_2, s_1, s_2, y_1, y_2 \geq 0
\end{align*}
\]

A basic feasible solution to this LP problem is $(x_1, x_2, s_1, s_2, y_1, y_2) = (0, 0, 0, 4, 3, 6)$, with cost 9.

Basic variables: $s_2 = 4, y_1 = 3, y_2 = 6$

nonbasic variables: $x_1, x_2, s_1$.

68
Notes

1. The auxiliary problem is always a minimization of \( \sum_{i=1}^{k} y_i \) whether the original problem is Minimization or Maximization. (Why?)

2. If \( \sum_{i=1}^{k} y_i > 0 \), the original LP problem is infeasible (Why?).

3. If \( \sum_{i=1}^{k} y_i = 0 \), then the original LP problem has a basic feasible solution.

A complete algorithm for LP problems in standard form.

Phase I

1. Introduce artificial variables \( y_1, y_2, \ldots, y_m \), wherever necessary, and apply the simplex method to the auxiliary problem with cost \( \sum_{i=1}^{m} y_i \).

2. If the optimal cost in the auxiliary problem is positive, the original problem is infeasible and the algorithm terminates.

3. If the optimal cost in the auxiliary problem is zero, a basic feasible solution to the original problem has been found as follows:

   (a) If no artificial variable is in the final basis, the artificial variables and the corresponding columns are eliminated, and a feasible basis for the original problem is available.

   (b) If the \( l \)th basic variable is an artificial variable, examine the \( l \)th entry of the columns \( B^{-1}A_j, j = 1, \ldots, n \).
      i. If the \( l \)th entry of the \( j \)th column is nonzero, apply a change of basis (with this entry serving as the pivot element): the \( l \)th basic variable exits and the \( x_j \) enters the basis.
      ii. Repeat this operation until all artificial variables are driven out of the basis.

Phase II

1. Let the final basis and tableau obtained from Phase I be the initial basis and tableau for Phase II.

2. Compute the reduced costs of all variables for the initial basis, using the cost coefficients of the original problem.

3. Apply the simplex method to the original problem.

Remark The purpose of phase I is to obtain a basic feasible solution to the original LP, if it exists.
Example 3.6 Use the 2-phase method to solve the LP problem:

Minimize \[ 4x_1 + x_2 \]
Subject to \[ 3x_1 + x_2 = 3 \]
\[ -4x_1 - 3x_2 \leq -6 \]
\[ x_1 + 2x_2 \leq 4 \]
\[ x_1, x_2 \geq 0. \]

Solution

Phase I. Auxiliary problem

Minimize \[ y_1 + y_2 \]
Subject to \[ 3x_1 + x_2 + y_1 = 3 \]
\[ 4x_1 + 3x_2 - s_1 + y_2 = 6 \]
\[ x_1 + 2x_2 + s_2 = 4 \]
\[ x_1, x_2, s_1, s_2, y_1, y_2 \geq 0. \]

By definition, \[ \bar{c}^T = c^T - c_B^T B^{-1} A. \]

For each \( i \), note that

\[ \bar{c}_i = c_i - c_B^T B^{-1} A_i = c_i - (c_{y_1}, c_{y_2}, c_{s_2}) B^{-1} A_i = c_i - (1, 1, 0) B^{-1} A_i. \]

Thus, the starting \( \bar{c} \)-row is obtained by applying row operations:

\[ \bar{c} - \text{row} = (c - \text{row}) - (y_1 - \text{row}) - (y_2 - \text{row}) - 0(s_2 - \text{row}). \]

Note that reduced costs of basic variables \( y_1, y_2, s_2 \) are zero.

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_1 )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( s_2 )</th>
<th>Soln</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>( c )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>( \bar{c} )</td>
<td>-7</td>
<td>-4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\( x_1 \) enters

<table>
<thead>
<tr>
<th></th>
<th>( y_1 )</th>
<th>3</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
</table>

\( y_1 \) leaves

<table>
<thead>
<tr>
<th></th>
<th>( y_2 )</th>
<th>4</th>
<th>3</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>6</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( s_2 )</th>
<th>1</th>
<th>2</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
</table>

(1)

\( \bar{c} \) enters

<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
<th>1</th>
<th>( \frac{2}{3} )</th>
<th>0</th>
<th>( \frac{2}{3} )</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
</table>

\( y_2 \) leaves

<table>
<thead>
<tr>
<th></th>
<th>( y_2 )</th>
<th>0</th>
<th>-1</th>
<th>( -\frac{2}{3} )</th>
<th>1</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( s_2 )</th>
<th>0</th>
<th>( -\frac{1}{3} )</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th></th>
</tr>
</thead>
</table>

(2)

\( \bar{c} \) enters

<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
</table>

\( x_2 \) enters

<table>
<thead>
<tr>
<th></th>
<th>( x_2 )</th>
<th>1</th>
<th>0</th>
<th>( \frac{1}{3} )</th>
<th>-( \frac{2}{3} )</th>
<th>( -\frac{1}{3} )</th>
<th>0</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( s_2 )</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
</table>

At the optimum, \( y_1 + y_2 = 0 \), thus the original problem has a basic feasible solution, namely \( (x_1, x_2, s_1, s_2) = (3/5, 6/5, 0, 1) \) (basic variables are \( x_1, x_2, s_2 \)) and we proceed to phase II.
Phase II. The artificial variables \((y_1 \text{ and } y_2)\) have now served their purpose and must be dispensed with in all subsequent computations (by setting them to be zero, i.e. \(y_1 = 0, y_2 = 0\)). In the simplex tableau, columns of \(y_1 \text{ and } y_2\) are removed.

With basic variables \(x_1, x_2, s_2\), we have

\[
\bar{c}_i = c_i - c^T_BB^{-1}A_i = c_i - (c_{x_1}, c_{x_2}, c_{s_2})B^{-1}A_i = c_i - (4, 1, 0)B^{-1}A_i.
\]

The starting \(\bar{c}\)-row for the simplex method can thus be obtained via applying row operations

\[
\bar{c}\text{-row} = (c\text{-row}) - 4 \times (x_1\text{-row}) - 1 \times (x_2\text{-row}).
\]

**Simplex Tableau**

<table>
<thead>
<tr>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>Soln</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\bar{c})</td>
<td>0</td>
<td>0</td>
<td>(-\frac{1}{5})</td>
<td>0</td>
<td>(-\frac{18}{5})</td>
</tr>
</tbody>
</table>

\(s_1\) enters

| \(x_1\) | 1      | 0      | \(\frac{1}{5}\) | 0      | \(\frac{3}{5}\) |
| \(x_2\) | 0      | 1      | \(-\frac{3}{5}\) | 0      | \(\frac{4}{5}\) |
| \(s_2\) | 0      | 0      | 1      | 1      | 1    |

\(s_2\) leaves

\[
\bar{c} = 0 \quad 0 \quad 0 \quad \frac{1}{5} \quad -\frac{17}{5}.
\]

\(\bar{c}\text{-row} = (c\text{-row}) - 4 \times (x_1\text{-row}) - 1 \times (x_2\text{-row})\)

optimum

| \(x_1\) | 1      | 0      | 0      | \(-\frac{1}{5}\) | \(-\frac{17}{5}\) |
| \(x_2\) | 0      | 1      | 0      | \(\frac{3}{5}\) | \(\frac{9}{5}\) |
| \(s_1\) | 0      | 0      | 1      | 1      | 1    |

Thus, the optimal solution is \((x_1, x_2) = (\frac{2}{5}, \frac{9}{5})\) with cost \(\frac{17}{5}\).

**Note** The artificial variables are removed in Phase II only when they are **nonbasic** at the end of Phase I. It is possible, however, that an artificial variable remains **basic** at zero level at the end of Phase I. In this case, provisions must be made to ensure that it never becomes positive during Phase II computations (refer to the algorithm).
Example 3.7 (Infeasible Solution.)

$$\begin{align*}
\text{Minimize} & \quad -3x_1 - 2x_2 \\
\text{Subject to} & \quad 2x_1 + x_2 \leq 2 \\
& \quad 3x_1 + 4x_2 \geq 12 \\
& \quad x_1, x_2 \geq 0
\end{align*}$$

Solution

Auxiliary LP problem:

$$\begin{align*}
\text{Minimize} & \quad y \\
\text{Subject to} & \quad 2x_1 + x_2 + s_1 = 2 \\
& \quad 3x_1 + 4x_2 - s_2 + y = 12 \\
& \quad x_1, x_2, s_1, s_2, y \geq 0
\end{align*}$$

\[
\begin{array}{c|cccc|c|cc}
\text{Basic} & x_1 & x_2 & s_1 & s_2 & y & \text{solution} \\
\hline
c & 0 & 0 & 0 & 0 & 1 & 0 \\
\bar{c} & -3 & -4 & 0 & 1 & 0 & -12 \\
\hline
x_2 \rightarrow x_1 & 2 & 1 & 1 & 0 & 0 & 2 \\
s_1 \leftarrow y & 3 & 4 & 0 & -1 & 1 & 12 \\
\bar{c} & 5 & 0 & 4 & 1 & 0 & -4 \\
\hline
x_2 & 2 & 1 & 1 & 0 & 0 & 2 \\
y & -5 & 0 & -4 & -1 & 1 & 4 \\
\end{array}
\]

The tableau is optimal but cost $4 \neq 0$, thus there is no feasible solution.
(b) The big-$M$ Method

Similar to the two-phase method, the big-$M$ method starts with the LP in the standard form, and augment an artificial variable $y_i$ for any constraint that does not have a slack. Such variables, together with slack variables, then become the starting basic variables.

We penalize each of these variables by assigning a very large coefficient ($M$) in the objective function:

Minimize objective function $+ \sum M y_i$, where $M > 0$ (minimization)

or

Maximize objective function $- \sum M y_i$, where $M > 0$ (maximization)

For sufficiently large choice of $M$, if the original LP is feasible and its optimal value is finite, all of the artificial variables are eventually driven to zero, and we have the minimization or maximization of the original objective function.

The coefficient $M$ is not fixed with any numerical value. It is always treated as a larger number whenever it is compared to another number. Thus the reduced costs are functions of $M$.

We apply simplex algorithms to the modified objective and the same constraints as in the Auxiliary LP problem in the 2-phase method.

**Example 3.8** Solve the LP problem by the big-$M$ method.

\[
\begin{align*}
\text{Minimize} & \quad 4x_1 + x_2 \\
\text{Subject to} & \quad 3x_1 + x_2 = 3 \\
& \quad 4x_1 + 3x_2 \geq 6 \\
& \quad x_1 + 2x_2 \leq 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

**Solution** The standard form of the LP:

\[
\begin{align*}
\text{Minimize} & \quad 4x_1 + x_2 \\
\text{Subject to} & \quad 3x_1 + x_2 = 3 \\
& \quad 4x_1 + 3x_2 - s_1 = 6 \\
& \quad x_1 + 2x_2 + s_2 = 4 \\
& \quad x_1, x_2, s_1, s_2 \geq 0
\end{align*}
\]

We augment two artificial variables $y_1$ and $y_2$ in the 1st and 2nd equations, and penalize $y_1$ and $y_2$ in the objective function by adding $M y_1 + M y_2$. 

73
The modified LP with its artificial variables becomes:

Minimize 4\(x_1\) + \(x_2\) + \(My_1\) + \(My_2\)
Subject to 3\(x_1\) + \(x_2\) + \(y_1\) = 3
4\(x_1\) + 3\(x_2\) − \(s_1\) + \(y_2\) = 6
\(x_1\) + 2\(x_2\) + \(s_2\) = 4
\(x_1, x_2, s_1, s_2, y_1, y_2 \geq 0\)

Choose artificial variables and slack variables to be basic variables.

Thus, \(x_B = (y_1, y_2, s_2)^T\).

Since \(c_B^T = (M, M, 0)\), we obtain the starting \(\bar{c}\)-row as follows:

Starting \(\bar{c}\)-row = \((c\text{-row}) - M \times (y_1\text{-row}) - M \times (y_2\text{-row}) - 0 \times (s_2\text{-row})\).

In tableau form, we have:

<table>
<thead>
<tr>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(s_1)</th>
<th>(y_1)</th>
<th>(y_2)</th>
<th>(s_2)</th>
<th>Soln</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>(M)</td>
<td>(M)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\bar{c})</td>
<td>4 - 7(M)</td>
<td>1 - 4(M)</td>
<td>(M)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(-9M)</td>
</tr>
</tbody>
</table>

\(x_1\) enters

\(y_1\) leaves

\(y_2\)

\(s_2\)

<table>
<thead>
<tr>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(s_1)</th>
<th>(y_1)</th>
<th>(y_2)</th>
<th>(s_2)</th>
<th>Soln</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>1</td>
<td>(\frac{3}{2})</td>
<td>0</td>
<td>(\frac{3}{2})</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(\bar{c})</td>
<td>0</td>
<td>(-\frac{3}{2} - \frac{3}{2}M)</td>
<td>(M)</td>
<td>(-\frac{3}{2} - \frac{3}{2}M)</td>
<td>0</td>
<td>0</td>
<td>(-4 - 2M)</td>
</tr>
</tbody>
</table>

\(x_2\) enters

\(y_2\) leaves

\(s_1\)

\(s_2\)

<table>
<thead>
<tr>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(s_1)</th>
<th>(y_1)</th>
<th>(y_2)</th>
<th>(s_2)</th>
<th>Soln</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>1</td>
<td>0</td>
<td>(\frac{3}{5})</td>
<td>(\frac{3}{5})</td>
<td>(-\frac{4}{5})</td>
<td>0</td>
<td>(\frac{4}{5})</td>
</tr>
<tr>
<td>(\bar{c})</td>
<td>0</td>
<td>1</td>
<td>(-\frac{3}{5})</td>
<td>(-\frac{4}{5})</td>
<td>(\frac{3}{5})</td>
<td>0</td>
<td>(\frac{4}{5})</td>
</tr>
</tbody>
</table>

Optimum

Therefore, the optimal solution is \((x_1, x_2) = (\frac{2}{5}, \frac{9}{5})\) with optimal cost \(\frac{17}{5}\).
Since it contains no artificial variables at positive level, the solution is feasible with respect to the original problem before the artificial variables are added.
(If the problem has no feasible solution, at least one artificial variable will be positive in the optimal solution).
4.4 Special Cases in Simplex Method Application

(A) Degeneracy
A basic feasible solution in which one or more basic variables are zero is called a degenerate basic feasible solution. A tie in the minimum ratio rule leads to the degeneracy in the solution. From the practical point of view, the condition reveals that the model has at least one redundant constraint at that basic feasible solution.

Example 4.1 (Degenerate Optimal Solution)

\[
\begin{align*}
\text{Minimize} & \quad -3x_1 - 9x_2 \\
\text{subject to} & \quad x_1 + 4x_2 \leq 8 \\
& \quad x_1 + 2x_2 \leq 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

<table>
<thead>
<tr>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(S_1)</th>
<th>(S_2)</th>
<th>Solution</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_2) enters</td>
<td>(S_1)</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>(S_1) leaves</td>
<td>(S_2)</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>(x_1) enters</td>
<td>(x_2)</td>
<td>(\frac{1}{4})</td>
<td>1</td>
<td>(\frac{1}{4})</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(S_2) leaves</td>
<td>(S_2)</td>
<td>(\frac{1}{2})</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(x_1) optimal</td>
<td>(x_2)</td>
<td>0</td>
<td>1</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
<td>2</td>
</tr>
</tbody>
</table>

Note: In iteration 2, the entering variable \(x_1\) replaces \(S_2\), where \(S_2 = 0\) is a basic variable, hence degeneracy remains in the optimum.

Looking at the graphical solution, we see that 3 lines pass through the optimum \((x_1 = 0, x_2 = 2)\). We need only 2 lines to identify a point in a two-dimensional problem hence we say that the point is overdetermined. For this reason, we conclude that one of the constraints is redundant. There are no reliable techniques for identifying redundant constraint directly from the tableau. In the absence of graphical representation, we may have to rely on other means to locate the redundancy in the model.
Theoretical Implications of Degeneracy

(a) The objective value is not improved (−18) in iterations 1 and 2. It is possible that the simplex iteration will enter a loop without reaching the optimal solution. This phenomenon is called “cycling”, but seldom happens in practice.

(b) Both iterations 1 and 2 yield identical values:

\[ x_1 = 0, \ x_2 = 2, \ S_1 = 0, \ S_2 = 0, \ z = 18 \]

but with different classifications as basic and nonbasic variables.

Question. Can we stop at iteration 1 (when degeneracy first appears even though it is not optimum? No, as we shall see in the next example.

Example 4.2. (Temporarily Degenerate Solution)

\[
\begin{align*}
\text{Minimize} & \quad -3x_1 - 2x_2 \\
\text{Subject to} & \quad 4x_1 + 3x_2 \leq 12 \\
& \quad 4x_1 + x_2 \leq 8 \\
& \quad 4x_1 - x_2 \leq 8 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>-3</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( S_1 ) leaves</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>12 3</td>
</tr>
<tr>
<td>( S_2 ) enters</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>8 2</td>
</tr>
<tr>
<td>( S_3 ) enters</td>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>8 2</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{Minimize} & \quad 0 \\
\text{Subject to} & \quad 4 - \frac{x_2}{3} - \frac{2}{3}x_2 \\
& \quad 4 - \frac{x_2}{3} - \frac{2}{3}x_2 \\
& \quad 4 - \frac{x_2}{3} - \frac{2}{3}x_2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>( S_1 ) enters</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>( S_2 ) leaves</td>
<td>1</td>
<td>\frac{1}{4}</td>
<td>0</td>
<td>\frac{1}{4}</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( S_3 ) leaves</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{Minimize} & \quad 0 \\
\text{Subject to} & \quad \frac{1}{3}x_2 - \frac{1}{3}x_2 - \frac{1}{3}x_2 \\
& \quad \frac{1}{3}x_2 - \frac{1}{3}x_2 - \frac{1}{3}x_2 \\
& \quad \frac{1}{3}x_2 - \frac{1}{3}x_2 - \frac{1}{3}x_2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>0</td>
<td>0</td>
<td>\frac{1}{3}</td>
<td>\frac{1}{3}</td>
<td>0</td>
<td>\frac{1}{3}</td>
</tr>
<tr>
<td>( S_1 ) enters</td>
<td>0</td>
<td>1</td>
<td>\frac{1}{2}</td>
<td>-\frac{1}{2}</td>
<td>0</td>
<td>\frac{1}{2}</td>
</tr>
<tr>
<td>( S_2 ) leaves</td>
<td>1</td>
<td>0</td>
<td>-\frac{1}{8}</td>
<td>\frac{3}{8}</td>
<td>0</td>
<td>\frac{3}{8}</td>
</tr>
<tr>
<td>( S_3 ) leaves</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Note The entering variable \( x_2 \) has a negative coefficient corresponding to \( S_3 \), hence \( S_3 \) cannot be the leaving variable. Degeneracy disappears in the final optimal solution.
(B) Alternative Optima

When the objective function is parallel to a binding constraint, the objective function will assume the same optimal value at more than one solution point. For this reason they are called alternative optima.

Example 4.3.

Minimize \(-2x_1 - 4x_2\)
Subject to \(x_1 + 2x_2 \leq 5\)
\(x_1 + x_2 \leq 4\)
\(x_1, x_2 \geq 0\)

<table>
<thead>
<tr>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(S_1)</th>
<th>(S_2)</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_2) enters (S_1)</td>
<td>(S_1)</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(S_1) leaves (S_2)</td>
<td>(S_2)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(\bar{c})</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

\(x_1\) enters \(S_2\)\):
\(S_2\) leaves \(\bar{c}\)\):
\(\bar{c}\) | 0 | 0 | 2 | 0 | 10 |

| \(x_2\) | 0 | 1 | 1 | -1 | 1 |
| \(x_1\) | 1 | 0 | -1 | 2 | 3 |

When the reduced cost of a nonbasic variable (here \(x_1\)) is zero, it indicates that \(x_1\) can be an entering basic variable without changing the cost value, but causing a change in the values of the variables. The family of alternative optimal solutions (basic and nonbasic) is given by: \((x_1, x_2) = \lambda(0, \frac{5}{2}) + (1 - \lambda)(3, 1)\), where \(0 \leq \lambda \leq 1\).

Remark If an LP problem has \(k\) \((k \geq 2)\) optimal basic feasible solutions: \(x_1, x_2, \ldots, x_k\), then LP problem has infinitely many optimal solutions and the general form of an optimal solution is \(\sum_{i=1}^{k} \lambda_i x_i\) where \(\sum_{i=1}^{k} \lambda_i = 1\) and \(\lambda_i \geq 0\) for \(i = 1, 2, \ldots, k\).
(C) Unbounded Solution

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraint and we have an **unbounded solution space**. It is not necessarily, however, that an unbounded solution space yields an unbounded value for the objective function. Unbounded objective value in a model indicates the model is poorly constructed - an infinite cost or profit!!

**General rule of detecting Unboundedness**

If at any iteration, the constraint coefficients $B^{-1}A_j$ of a nonbasic variable $x_j$ are all nonpositive, the solution space is unbounded in that direction. If, in addition, the reduced cost $\bar{c}_j$ of that nonbasic variable is negative (respectively positive) in the minimization (respectively maximization) problem, then the objective value is also unbounded.

**Example 4.4** (Unbounded Objective Value)

Minimize $-2x_1 - x_2$

Subject to $x_1 - x_2 \leq 10$

$2x_1 \leq 40$

$x_1, x_2 \geq 0$

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_1$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>40</td>
</tr>
</tbody>
</table>

Note that $x_2$ is a candidate for entering the solution. All the constraint coefficients in $x_2$-column are zero or negative implying that $x_2$ can be increased indefinitely without violating any of the constraints. Therefore, the solution space is unbounded in the $x_2$-direction and the LP has no bounded optimal solution because $x_2$ is a candidate of being entering variable.

**Example 4.5.** (Unbounded Solution Space but Finite Optimal Objective Value)

Minimize $-6x_1 + 2x_2$

Subject to $2x_1 - x_2 \leq 2$

$x_1, x_2 \leq 4$

$x_1, x_2 \geq 0$

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>-6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_1$</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$S_2$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>$c$</td>
<td>0</td>
<td>-1</td>
<td>3</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>-1/2</td>
<td>1/2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$S_2$</td>
<td>0</td>
<td>1/2</td>
<td>-1/2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$c$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

78
(D) Infeasible Solution

If the constraints cannot be satisfied simultaneously, the model is said to have no feasible solution. This situation can never occur if all the constraints are of the type \( \leq \), since the slack variables always provide a feasible solution. When we have constraints of other types, we introduce artificial variables, which, by their very design, do not provide a feasible solution to the original model if \( y_i \neq 0 \) in the optimal solution in the big M-technique, and if \( \sum y_i > 0 \) in Phase I in the two-phase method. From the practical point of view, an infeasible solution space shows that the model is not formulated correctly.

Example 4.6 Show that the following LP problem has no feasible solution.

\[
\begin{align*}
\text{Minimize} & \quad 3x_1 \\
\text{Subject to} & \quad 2x_1 + x_2 \geq 6 \\
& \quad 3x_1 + 2x_2 = 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Solution. We use the Big-M method. Adding artificial variables, we obtain:

\[
\begin{align*}
\text{Minimize} & \quad 3x_1 + My_1 + My_2 \\
\text{Subject to} & \quad 2x_1 + x_2 - s_1 + y_1 = 6 \\
& \quad 3x_1 + 2x_2 + y_2 = 4 \\
& \quad x_1, x_2, s_1, y_1, y_2 \geq 0
\end{align*}
\]

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_1 )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>R. H. S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>3 - 5M</td>
<td>-3M</td>
<td>M</td>
<td>M</td>
<td>0</td>
<td>-10M</td>
</tr>
<tr>
<td>( x_1 ) enters</td>
<td>( y_1 )</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( y_2 ) leaves</td>
<td>( y_2 )</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( c )</td>
<td>0</td>
<td>( \frac{(M-6)}{3} )</td>
<td>M</td>
<td>0</td>
<td>( \frac{(M-3)}{3} )</td>
<td>( \frac{10}{3} M + 4 )</td>
</tr>
<tr>
<td>optimum</td>
<td>( x_1 )</td>
<td>0</td>
<td>( \frac{2}{3} )</td>
<td>-1</td>
<td>1</td>
<td>( \frac{2}{3} )</td>
</tr>
</tbody>
</table>

This is an optimal tableau. However, the artificial variable \( y_1 = \frac{10}{3} \), which is positive, and hence the original problem has no feasible solution.

Exercise Use the phase 1 of the 2-phase method to show that the LP problem has no feasible solution.
Chapter 5

Duality Theory

Starting with a linear programming problem, called the primal LP, we introduce another linear programming problem, called the dual problem. Duality theory deals with the relation between these two LP problems. It is also a powerful theoretical tool that has numerous applications, and leads to another algorithm for linear programming (the dual simplex method).

Motivation

Consider the standard form problem

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}
\]

which we call the primal problem. Let \( x^* \) be an optimal solution, assumed to exist. We introduce a relaxed problem

\[
\begin{align*}
\text{minimize} & \quad c^T x + p^T (b - Ax) \\
\text{subject to} & \quad x \geq 0,
\end{align*}
\]

in which the constraint \( Ax = b \) is replaced by a penalty \( p^T (b - Ax) \), where \( p \) is a vector of the same dimension as \( b \).

Let \( g(p) \) be the optimal cost for the relaxed problem, as a function of \( p \). Thus,

\[
g(p) \leq c^T x^* + p^T (b - Ax^*) = c^T x^*.
\]

This implies that each \( p \) leads to a lower bound \( g(p) \) for the optimal cost \( cx^* \).

The problem

\[
\begin{align*}
\text{maximize} & \quad g(p) \\
\text{subject to} & \quad \text{No constraints}
\end{align*}
\]

which searches for the greatest lower bound, is known as the dual problem.
Note:

1. \( g(p) = \min_{x \geq 0} \left[ cx + p^T (b - Ax) \right] = p^T b + \min_{x \geq 0} \left[ (c^T - p^T A) x \right] \).

2. \( \min_{x \geq 0} (c^T - p^T A) x = \begin{cases} 0, & \text{if } c^T - p^T A \geq 0, \\ -\infty, & \text{otherwise.} \end{cases} \)

Thus, the dual problem is the same as the linear programming problem

\[
\begin{align*}
\text{maximize} & \quad p^T b \\
\text{subject to} & \quad p^T A \leq c^T.
\end{align*}
\]

### 5.1 The dual problem.

As motivated by the observation in the previous section, we define the dual problem of a (primal) LP problem as follows.

**Definition 1.1:** Given a (primal) LP problem

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}
\]

the associated dual LP problem is

\[
\begin{align*}
\text{maximize} & \quad p^T b \\
\text{subject to} & \quad p^T A \leq c^T.
\end{align*}
\]

We only define the dual problem for the standard LP problem. LP problems may appear in various forms. We will derive their dual problems in the following steps: (1) Convert the original (primal) LP problem to a standard LP problem; (2) Formulate the dual problem of the standard LP problem by using Definition 1.1; (3) Simplify the dual problem, if necessary.

**Example 1.1**

Consider the primal problem:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b, \\
& \quad x \text{ free}
\end{align*}
\]

Introducing surplus variables and replacing \( x \) by sign-constrained variables in the original primal problem yield the following equivalent LP:

\[
\begin{align*}
\text{minimize} & \quad c^T x^+ - c^T x^- \\
\text{subject to} & \quad Ax^+ - Ax^- - s = b, \\
& \quad x^+ \geq 0, x^- \geq 0, s \geq 0.
\end{align*}
\]
By Definition 1.1, the dual problem of this standard LP is

\[
\begin{align*}
\text{maximize} & \quad p^T b \\
\text{subject to} & \quad p^T A \leq c^T \\
& \quad -p^T A \leq -c^T \\
& \quad -p^T I \leq 0.
\end{align*}
\]

Note that \( p^T A = c^T \) is equivalent to \( p^T A \leq c^T \) and \( -p^T A \leq -c^T \).
Thus, the dual obtained here is equivalent to

\[
\begin{align*}
\text{maximize} & \quad p^T b \\
\text{subject to} & \quad p \geq 0 \\
& \quad p^T A = c^T.
\end{align*}
\]

The above shows the following pair of primal and dual LPs:

**Primal**
- minimize \( c^T x \)
- subject to \( Ax \geq b, \ x \text{ free} \)

**Dual**
- maximize \( p^T b \)
- subject to \( p \geq 0, \ p^T A = c^T \)

In general, we can show the pair of primal and dual problems are related as follows. Let \( A \) be a matrix with rows \( a_i^T \) and columns \( A_j \).

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \geq b_i, \ i \in M_+ \\
& \quad a_i^T x \leq b_i, \ i \in M_- \\
& \quad a_i^T x = b_i, \ i \in M_0 \\
& \quad x_j \geq 0, \ j \in N_+ \\
& \quad x_j \leq 0, \ j \in N_- \\
& \quad x_j \text{ free}, \ j \in N_0 \\
\text{maximize} & \quad p^T b \\
\text{subject to} & \quad p_i \geq 0, \ i \in M_+ \\
& \quad p_i \leq 0, \ i \in M_- \\
& \quad p_i \text{ free}, \ i \in M_0 \\
& \quad p^T A_j \leq c_j, \ j \in N_+ \\
& \quad p^T A_j \geq c_j, \ j \in N_- \\
& \quad p^T A_j = c_j, \ j \in N_0.
\end{align*}
\]

Notes:
1. For each functional constraint \( a_i^T x (\geq, \leq, =) b_i \) in the primal, we introduce a variable \( p_i (\geq 0, \leq 0, \text{ free}) \) respectively in the dual problem.
2. For each variable \( x_j (\geq 0, \leq 0, \text{ free}) \) in the primal problem, there is a corresponding constraint \( (\leq, \geq, =) c_j \) respectively in the dual problem.

In summary:

<table>
<thead>
<tr>
<th>constraints</th>
<th>minimize</th>
<th>maximize</th>
<th>variables</th>
<th>constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \geq )</td>
<td>( \geq 0 )</td>
<td>( \leq 0 )</td>
<td>( \geq )</td>
<td>( = )</td>
</tr>
<tr>
<td>( \leq )</td>
<td>( \leq 0 )</td>
<td>( \geq 0 )</td>
<td>( \leq )</td>
<td>( = )</td>
</tr>
<tr>
<td>( = )</td>
<td>( = )</td>
<td>( = )</td>
<td>( = )</td>
<td>( = )</td>
</tr>
<tr>
<td>\text{free}</td>
<td>( \text{free} )</td>
<td>( \text{free} )</td>
<td>( \text{free} )</td>
<td>( \text{free} )</td>
</tr>
</tbody>
</table>
Indeed, which side in the table is regarded as primal and which as dual does not matter, because we can show (exercise)

**Theorem (The dual of the dual is the primal.)**

If we transform the dual into an equivalent minimization problem, and then form its dual, we obtain a problem equivalent to the original problem.

**Example 1.2** Consider the following primal problem:

\[
\begin{align*}
\text{minimize} & \quad x_1 + 2x_2 + 3x_3 \\
\text{subject to} & \quad -x_1 + 3x_2 = 5 \\
& \quad 2x_1 - x_2 + 3x_3 \geq 6 \\
& \quad x_3 \leq 4 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \leq 0 \\
& \quad x_3 \text{ free}
\end{align*}
\]

(a) Write down the dual problem.

(b) Verify that the primal problem and dual of dual obtained are equivalent.
5.2 The duality theorem.

Theorem (Weak duality theorem)
In a primal-dual pair, the objective value of the maximization problem is smaller than or equal to the objective value of the minimization problem. That is, for a minimization (respectively maximization) primal LP, if $x$ is a feasible solution to the primal problem and $p$ is a feasible solution to the dual problem, then

$$p^Tb \leq c^Tx \quad \text{(respectively } p^Tb \geq c^Tx).$$

Proof. We prove the result for a minimization primal LP.

For any vectors $x$ and $p$, define

$$u_i = p_i(a_i^Tx - b_i),$$
$$v_j = (c_j - p^TA_j)x_j.$$

Refer to the definition of the dual problem, note that both $p_i$ and $(a_i^Tx - b_i)$ are of the same sign or one of them is zero; and both $(c_j - p^TA_j)$ and $x_j$ are of the same sign or one of them is zero.

Suppose that $x$ and $p$ are primal and dual feasible solutions, respectively. Then,

$$u_i \geq 0, \forall i,$$
$$v_j \geq 0, \forall j.$$

Thus, we have

$$\sum_i u_i = \sum_i p_i(a_i^Tx - b_i) = p^TAx - p^Tb$$
and

$$\sum_j v_j = \sum_j (c_j - p^TA_j)x_j = c^Tx - p^TAx.$$

Adding both equalities yields the required inequality:

$$0 \leq \sum_i u_i + \sum_j v_j = c^Tx - p^Tb.$$
Example 2.01 Consider the following linear programming problem.

\[
\begin{align*}
\text{Minimize} & \quad -3x_1 - 2x_2 \\
\text{Subject to} & \quad x_1 + 2x_2 \leq 6 \\
& \quad 2x_1 + x_2 \leq 8 \\
& \quad -x_1 + x_2 \leq 1 \\
& \quad x_2 \leq 2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Note that \((x_1, x_2) = (1, 1)\) is a primal feasible solution, with objective value \(-5\), whereas \((p_1, p_2, p_3, p_4) = (-1, 0, 0, 4)\) is a dual feasible solution with objective value \(-14\).

This verifies the weak duality theorem, i.e. the objective value of the maximization problem \(\leq\) the objective value of the minimization problem in a primal-dual pair.

The above pair of primal and dual objective values can be used to provide a range for the optimal value of the primal (and hence the dual) problem, i.e.

\[-14 \leq \text{the optimal objective value} \leq -5.\]

In fact, the optimal value is \(-12\frac{2}{3}\).

Corollary 1
Unboundedness in one problem implies infeasibility in the other problem.
If the optimal value in the primal (respectively dual) problem is unbounded, then the dual (respectively) problem must be infeasible.

Corollary 2
Let \(x\) and \(p\) be primal and dual feasible solutions respectively, and suppose that \(p^Tb = c^Tx\). Then, \(x\) and \(p\) are optimal primal and dual solutions respectively.
Theorem (Strong Duality)
If a linear programming problem has an optimal solution, so does its dual, and the respective optimal costs are equal.

Proof. Consider the standard form minimization primal problem and its dual problem:

Primal
minimize $c^T x$
subject to $Ax = b$
$x \geq 0$,

Dual
maximize $p^T b$
subject to $p^T A \leq c^T$
$p$ free,

Let $x$ be a primal optimal solution obtained from simplex method, with associated optimal basis $B$. Then $x_B = B^{-1}b$ is the corresponding vector of basic variables and $c_B$ is the vector of the costs of the basic variables.

Note that at optimal:

$$c^T - c_B^T B^{-1} A \geq 0.$$ 

Now, define $p^T = c_B^T B^{-1}$. (Aim: Show: $p$ is dual optimal.)
Then $p^T A \leq c^T$, showing that $p$ is dual feasible.

Moreover, $p^T b = c_B^T B^{-1} b = c_B^T x_B = c^T x$.

Thus, by Corollary 2, $p$ is dual optimal, and the optimal dual cost is equal to the optimal primal cost.

Remark From the proof, we note that, for a standard form LP problem, if $x$ is a primal optimal solution, with associated basis matrix $B$ and $c_B$ is the vector of the costs of the basic variables, then the dual optimal solution is given by

$$p^T = c_B^T B^{-1}.$$
Example 2.1 Consider the LP problem (cf: Example 3.1 (IV))

\[
\begin{align*}
\text{Minimize} & \quad -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 \\
\text{Subject to} & \quad x_1 + 2x_2 + s_1 = 6 \\
& \quad 2x_1 + x_2 + s_2 = 8 \\
& \quad -x_1 + x_2 + s_3 = 1 \\
& \quad x_2 + s_4 = 2 \\
& \quad x_1, x_2, s_1, s_2, s_3, s_4 \geq 0
\end{align*}
\]

with optimal solution \((x_1, x_2, s_1, s_2, s_3, s_4) = (10/3, 4/3, 0, 0, 3, 2/3)\), where \(x_B = (x_2, x_1, s_3, s_4)\).

Thus the dual optimal solution is \(p^T = c_B^T B^{-1}\)

\[
= \begin{bmatrix}
-2 & -3 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\frac{2}{3} & -\frac{1}{3} & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 \\
-\frac{1}{3} & \frac{1}{3} & 1 & 1 \\
-\frac{2}{3} & \frac{1}{3} & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
-1/3 & -4/3 & 0 & 0 \\
\end{bmatrix}.
\]

Alternatively, one can obtain an optimal dual solution \(p\) from the optimal (primal) simplex tableau readily if the starting basis \(B_0 = I\).

The vector of reduced costs \(\bar{c}_{B_0}\) of starting basic variables \(x_{B_0}\) in the optimal tableau with \(B\) is

\[
\bar{c}_{B_0}^T = c_{B_0}^T - c_B^T B^{-1} B_0 = c_{B_0}^T - c_B^T B^{-1} = c_{B_0}^T - p^T.
\]

Thus, an optimal dual solution is \(p^T = c_{B_0}^T - \bar{c}_{B_0}^T\).
**Example 2.2 (a)** Consider the LP problem (cf: Example 3.1 (IV))

Minimize 
\[-3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4\]
Subject to 
\[x_1 + 2x_2 + s_1 = 6\]
\[2x_1 + x_2 + s_2 = 8\]
\[-x_1 + x_2 + s_3 = 1\]
\[x_2 + s_4 = 2\]
\[x_1, x_2, s_1, s_2, s_3, s_4 \geq 0\]

with optimal tableau as follows:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{c}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{4}{3}$</td>
<td>0</td>
<td>0</td>
<td>12 $\frac{4}{3}$</td>
</tr>
<tr>
<td>Optimum</td>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>$\frac{2}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>$-\frac{1}{3}$</td>
<td>$\frac{2}{3}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{10}{3}$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\frac{3}{3}$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>0</td>
<td>0</td>
<td>$-\frac{2}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>0</td>
<td>1</td>
<td>$\frac{2}{3}$</td>
</tr>
</tbody>
</table>

With $x_{B_0} = (s_1, s_2, s_3, s_4)$, the starting $B_0 = I$, and $c_{B_0}^T = (0, 0, 0, 0)$.

From the optimal tableau, $\bar{c}_{B_0}^T = (\frac{1}{3}, \frac{4}{3}, 0, 0)$.

Thus, the optimal dual solution $p^T = c_{B_0}^T - \bar{c}_{B_0}^T = (-\frac{1}{3}, -\frac{4}{3}, 0, 0)$.

**Example 2.2 (b)** From Example 3.8 (Chapter IV), the modified LP with its artificial variables is

Minimize 
\[4x_1 + x_2 + My_1 + My_2\]
Subject to 
\[3x_1 + x_2 + y_1 = 3\]
\[4x_1 + 3x_2 - s_1 + y_2 = 6\]
\[x_1 + 2x_2 + s_2 = 4\]
\[x_1, x_2, s_1, s_2, y_1, y_2 \geq 0\]

The $\bar{c}$-row in the optimal tableau is:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$s_2$</th>
<th>Soln</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{c}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-\frac{7}{5} + M$</td>
<td>$M$</td>
<td>$\frac{1}{5}$</td>
<td>$-\frac{15}{5}$</td>
</tr>
</tbody>
</table>

Starting basic variables: $y_1, y_2$ and $s_2$, with corresponding cost coefficient: $M$, $M$, and 0 respectively. Thus,

$p^T = (M, M, 0) - (-\frac{7}{5} + M, M, \frac{1}{5}) = (\frac{7}{5}, 0, -\frac{1}{5})$. 

88
The complementary slackness conditions in the next theorem provides a useful relation between optimal primal and dual solutions. Given an optimal solution to one problem, we can use these conditions to find the optimal solution of the other LP.

**Theorem** (Complementary Slackness Theorem.)

Let \( x \) and \( p \) be feasible solutions to the primal problem and dual problem respectively. The vectors \( x \) and \( p \) are optimal solutions for the two respective problems if and only if

\[
\begin{align*}
p_i(a^T_i x - b_i) &= 0, \quad \forall i \\
(c_j - p^T A_j) x_j &= 0, \quad \forall j.
\end{align*}
\]

**Proof.** In the proof of Weak Duality Theorem, we have

\[
\begin{align*}
u_i &= p_i(a^T x - b_i), \\
v_j &= (c_j - p^T A_j)x_j.
\end{align*}
\]

Assume that the primal is a minimization problem.

For \( x \) primal feasible and \( p \) dual feasible, we have \( u_i \geq 0 \) and \( v_j \geq 0 \) for all \( i \) and \( j \).

Moreover, we have

\[
c^T x - p^T b = \sum_i u_i + \sum_j v_j.
\]

By the Strong Duality Theorem, if \( x \) and \( p \) are optimal solutions for the two respective problems, then \( c^T x = p^T b \). Hence, \( \sum_i u_i + \sum_j v_j = 0 \) which implies that \( u_i = 0 \) and \( v_j = 0 \) for all \( i \) and \( j \).

Conversely, if \( u_i = 0 \) and \( v_j = 0 \) for all \( i \) and \( j \), then \( c^T x - p^T b = 0 \), i.e. \( c^T x = p^T b \).

By Corollary 2, both \( x \) and \( p \) are optimal. QED.

**Remark** Complementary slackness optimality conditions.

\[
\begin{align*}
u_i &= p_i(a^T_i x - b_i) = 0, \quad \forall i \\
v_j &= (c_j - p^T A_j)x_j = 0, \quad \forall j.
\end{align*}
\]
Example 2.3
Consider a problem in standard and its dual:

\[
\begin{align*}
\text{minimize} & \quad 13x_1 + 10x_2 + 6x_3 \\
\text{subject to} & \quad 5x_1 + x_2 + 3x_3 = 8 \\
& \quad 3x_1 + x_2 \leq 3 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{maximize} & \quad 8p_1 + 3p_2 \\
\text{subject to} & \quad 5p_1 + 3p_2 \leq 13 \\
& \quad p_1 + p_2 \leq 10 \\
& \quad 3p_1 \leq 6 \\
& \quad p_1 \text{ free, } p_2 \geq 0
\end{align*}
\]

(a) Verify that \(x^* = (1, 0, 1)^T\) is a solution to the primal problem.

(b) Use Complementary Slackness Theorem to verify that \(x^* = (1, 0, 1)^T\) is an optimal solution to the primal problem, and obtain a dual optimal solution.

Solution

(a) \(x^* = (x_1, x_2, x_3)^T = (1, 0, 1)^T\) is primal feasible. (Exc.)

(b) Suppose \(p = (p_1, p_2)^T\) is a dual feasible solution.

By the Complementary Slackness Theorem, both \(x^* = (x_1, x_2, x_3)^T = (1, 0, 1)^T\) and \(p = (p_1, p_2)^T\) are primal and dual optimal solutions if and only if the Complementary Slackness Optimality conditions are satisfied, i.e.

\[
\begin{align*}
u_1 &= 0, u_2 = 0, v_1 = 0, v_2 = 0 \quad \text{and} \quad v_3 = 0.
\end{align*}
\]

We shall find \((p_1, p_2)\) that satisfies the above conditions.

Note that \(u_1 = p_1(5x_1 + x_2 + 3x_3 - 8) = p_1(0) = 0\) and \(u_2 = p_1(3x + x_2 - 3) = p_2(0) = 0\). From the remaining conditions, we have

\[
\begin{align*}
v_1 &= x_1(13 - (5p_1 + 3p_2)) = (1)(13 - (5p_1 + 3p_2)) = 0 \\
v_2 &= x_2(10 - (p_1 + p_2)) = (0)(10 - (p_1 + p_2)) = 0 \\
v_3 &= x_3(6 - (3p_1)) = (1)(6 - (3p_1)) = 0
\end{align*}
\]

Thus, \(13 - (5p_1 + 3p_2) = 0\) and \(6 - 3p_1 = 0\). Solving yields \((p_1, p_2) = (2, 1)\).

Note that \((p_1 + p_2) = 3 \leq 10\), and \(p_2 = 2 \geq 0\). Thus, \((p_1, p_2) = (2, 1)\) is dual feasible since it satisfies all dual constraints.

Since \(x^* = (1, 0, 1)^T\) and \(p = (2, 1)^T\) satisfy the complementary slackness optimality conditions, by the Complementary Slackness Theorem, \(x^* = (x_1, x_2, x_3)^T = (1, 0, 1)^T\) and \(p = (p_1, p_2)^T = (2, 1)^T\) are optimal solutions to the two respective problems.
Example 2.4 Consider the following LP:

Minimize \( 8x_1 + 6x_2 - 10x_3 + 20x_4 - 2x_5 \)
Subject to \( 2x_1 + x_2 - x_3 + 2x_4 + x_5 = 25 \)
\( 2x_1 + 2x_3 - x_4 + 3x_5 = 20 \)
\( x_1, x_2, x_3, x_4, x_5 \geq 0 \)

Is \((x_1, x_2, x_3, x_4, x_5) = (10, 5, 0, 0, 0)\) an optimal solution to the above LP?

Solution Firstly, check \((x_1, x_2, x_3, x_4, x_5) = (10, 5, 0, 0, 0)\) is primal feasible.

(Exc.) Note that \(u_1 = u_2 = 0\) because of equality constraints.

Associating the dual variables \(p_1\) and \(p_2\) to the two constraints, the dual problem is:

Maximize \( 25p_1 + 20p_2 \)
Subject to \( 2p_1 + 2p_2 \leq 8 \)
\( p_1 \leq 6 \)
\( -p_1 + 2p_2 \leq -10 \)
\( 2p_1 - p_2 \leq 20 \)
\( p_1 + 3p_2 \leq -2 \)
\( p_1, p_2 \) unrestricted.

Suppose the feasible solution is optimal and \((p_1, p_2)\) is a dual optimal solution. By the Complementary Slackness Optimality Conditions, we must have

\[
\begin{align*}
v_1 &= x_1(8 - (2p_1 + 2p_2)) = 0 \\
v_2 &= x_2(6 - p_1) = 0 \\
v_3 &= x_3(-10 - (-p_1 + 2p_2)) = 0 \\
v_4 &= x_4(20 - (2p_1 - p_2)) = 0 \\
v_5 &= x_5(-2 - (p_1 + 3p_2)) = 0.
\end{align*}
\]

Since \(x_1 = 10 > 0\) and \(x_2 = 5 > 0\), we have \(8 - (2p_1 + 2p_2) = 0\) and \(6 - p_1 = 0\), i.e. \((p_1, p_2) = (6, -2)\).

Now we check for dual feasibility: it remains to check the last three dual constraints at \((p_1, p_2) = (6, -2)\):

\[-p_1 + 2p_2 = -10, 2p_1 - p_2 = 14 \leq 20, \text{ and } p_1 + 3p_2 = 0 > -2.\]
Thus the last dual constraint is not satisfied, and we conclude that \((p_1, p_2) = (6, -2)\) is not dual feasible.

Therefore, there is no dual feasible solution satisfying Complementary Slackness Optimality Conditions together. Hence, \((x_1, x_2, x_3, x_4, x_5) = (10, 5, 0, 0, 0)\) is not an optimal solution.
5.3 Economic interpretation of optimal dual variables.

At the optimal solution of both the primal and dual, there is an economic interpretation of the dual variables $p_i$ as marginal costs for a minimization primal problem, or as marginal profits for a maximization primal problem.

Consider the standard form problem and its dual problem:

Primal: minimize $c^T x$
subject to $Ax = b$
$x \geq 0,$

Dual: maximize $p^T b$
subject to $p^T A \leq c^T$
p free,

where $A$ is $m \times n$ with linearly independent rows.

Let $x^*$ be a nondegenerate primal optimal solution, with associated optimal basis $B$ and the corresponding dual optimal solution $p^*$ is given by $p^{*T} = c_B^T B^{-1}$.

Let $\Delta = (\Delta_1, \ldots, \Delta_i, \ldots, \Delta_n)^T$ where each $\Delta_i$ is a small change in $b_i$, for each $i$, such that $B^{-1}(b + \Delta) \geq 0$ (feasibility is maintained).

Note that $\bar{c}^T = c^T - c_B^T B^{-1} A \geq 0$ remain unaffected, and hence optimality conditions are not affected. Thus, $B^{-1}(b + \Delta)$, with the same basis matrix $B$, is an optimal solution to the perturbed problem (perturb means small change).

The optimal cost in the perturbed problem is

$$c_B^T B^{-1}(b + \Delta) = p^{*T}(b + \Delta) = p^{*T}b + p^{*T}\Delta = c_B^T B^{-1}b + p^{*T}\Delta.$$  

Thus, a small change $\Delta$ in $b$ results in a change of $p^{*T}\Delta$ in the optimal cost. In particular, for a fixed $i$, if $\Delta_i = \delta$, and $\Delta_j = 0$ for $j \neq i$, then the change in the optimal objective value is

$$p^{*T}\Delta = \delta p_i.$$  

Therefore, each component $p_i$ of the optimal dual vector $p^*$ indicates the contribution of $i$th requirement $b_i$ towards the objective function. Thus, $p_i$ is interpreted as the marginal cost (or shadow cost) of the $i$th requirement $b_i$.

**Remark**

For a maximization primal problem, the component $p_i$ of the optimal dual vector $p$ is interpreted as the marginal profit (or shadow price) per unit increase of the $i$th requirement $b_i$. It is also known as the worth of the resource or the $i$-th requirement $b_i$. 

92
Dual variables $p_i$’s can be used to rank the ‘requirements’ according to their contribution to the objective value. For example, in a minimization problem, if $p_1 < 0$, then increasing $b_1$ (sufficiently) will reduce the total cost. Thus, if $p_1 < 0$ and $p_2 < 0$, and we are allowed to increase only one requirement, then the requirement $b_i$ corresponds to the most negative $p_i$ is given a higher priority to increase.

**Example 3.1** (An example to illustrate the use of $p_i$.)

Consider the product-mix problem in which each of three products is processed on three operations. The limits on the available time for the three operations are 430, 460, 420 minutes daily and the profits per unit of the three products are $3, $2 and $5. The times in minutes per unit on the three operations are given as follows:

<table>
<thead>
<tr>
<th></th>
<th>Product 1</th>
<th>Product 2</th>
<th>Product 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Operation 1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Operation 2</td>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Operation 3</td>
<td>1</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

The LP model is written as:

Maximize $3x_1 + 2x_2 + 5x_3$ (daily profit)
Subject to $x_1 + 2x_2 + x_3 \leq 430$ (unit on usage of op. 1)
$3x_1 + 2x_3 \leq 460$ (unit on usage of op. 2)
$x_1 + 4x_2 \leq 420$ (unit on usage of op. 3)
$x_1, x_2, x_3 \geq 0$.

Adding slack variables $S_1, S_2$ and $S_3$ to the three constraints, the optimal tableau is given as:

<table>
<thead>
<tr>
<th></th>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$c$</td>
<td>$-4$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$-2$</td>
<td>$0$</td>
<td>$-1350$</td>
</tr>
<tr>
<td></td>
<td>$x_2$</td>
<td>$-1/4$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1/2$</td>
<td>$-1/4$</td>
<td>$0$</td>
<td>$100$</td>
</tr>
<tr>
<td></td>
<td>$x_3$</td>
<td>$3/2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1/2$</td>
<td>$0$</td>
<td>$230$</td>
</tr>
<tr>
<td></td>
<td>$S_3$</td>
<td>$2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-2$</td>
<td>$1$</td>
<td>$1$</td>
<td>$20$</td>
</tr>
</tbody>
</table>

(a) Suppose an additional minute for Operation 2 costs $1.50$, is it advisable to increase the limit of available time for Operation 2?

(b) Rank the three operations in order of priority for increase in time allocation.

**Solution.** The dual prices are found to be $p_1 = 1, p_2 = 2$ and $p_3 = 0$.

(a) $p_2 = 2$ implies that a unit (i.e. 1 minute) increase in the time Operation 2 causes an increase of $2$ in the objective value. Thus, for an increase of 1 hour, this would increase the objective value by $120$. Since the cost
of an additional minute for Operation 2 costs $1.50. There is a net profit of $0.50 for when we increase the time for Operation 2. It is advisable to increase the operation time for Operation 2.

(b) From the dual prices, $p_1 = 1$, $p_2 = 2$ and $p_3 = 0$, if we are to increase the limits on the available time for the three operations, we would give a higher priority to Operation 2 followed by Operation 1. Note that since $p_3 = 0$, increasing the limit on the available time for Operation 3 has no effect on the profit.

5.4 The dual Simplex Method.

Note that the primal optimality condition for a minimization LP:

$$c^T - c_B^T B^{-1} A \geq 0$$

is the same as the dual feasibility condition, with dual variable $p^T = c_B^T B^{-1}$.

The simplex method is an algorithm that maintains primal feasibility and works towards dual feasibility (i.e. primal optimality). A method with this property is generally called a *primal* algorithm. An alternative is to start with a dual feasible solution and work towards primal feasibility. This method is called a *dual* algorithm. We shall implement the dual simplex method in terms of the simplex tableau.

Assume that the rows of the matrix $A$ are linearly independent. Let $B$ be a basis matrix, consisting of $m$ linearly independent columns of $A$.

Consider the corresponding simplex tableau:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$c^T - c_B^T B^{-1} A$</td>
<td>$-c_B^T B^{-1} b$</td>
</tr>
<tr>
<td>$x_B$</td>
<td>$B^{-1} A$</td>
<td>$B^{-1} b$</td>
</tr>
</tbody>
</table>

In detail, ($\bar{c}_j = c_j - c_B^T B^{-1} A_j$)

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$\cdots$</th>
<th>$x_n$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$\bar{c}_1$</td>
<td>$\cdots$</td>
<td>$\bar{c}_n$</td>
<td>$-c_B^T x_B$</td>
</tr>
<tr>
<td>$x_B(1)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_B(i)$</td>
<td>$B^{-1} A_1$</td>
<td>$\cdots$</td>
<td>$B^{-1} A_n$</td>
<td>$B^{-1} b$</td>
</tr>
<tr>
<td>$x_B(m)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
An iteration of the dual simplex method.

1. For a minimization (respectively maximization) problem, a typical iteration starts with the tableau associated with a basis matrix $B$ and with all reduced costs nonnegative (respectively nonpositive).

2. Examine the components of the vectors $x_B = B^{-1}b$.
   If they are all nonnegative, we have an optimal basic feasible solution and the algorithm stops;
   Else, choose some $l$ such that $x_{B(l)} < 0$.

3. Consider the $l$-th row (the pivot row) of the tableau, with elements $v_1, v_2, \ldots, v_n$.
   If $v_i \geq 0$ for all $i$, then the primal LP is infeasible and the algorithm stops.
   Else, for each $i$ such that $v_i < 0$, compute the ratio $|\bar{c}_i/v_i|$ and let $j$ be the index of a column that corresponds to the smallest ratio. The column $A_{B(l)}$ leaves the basis and the column $A_j$ enters the basis. (The minimum ratio ensures that the optimality conditions is maintained.)

4. Add to each row of the tableau a multiple of the $l$-row (the pivot row) so that $v_j$ (the pivot element) becomes 1 and all other entries of the pivot column become 0.

Key points to note

1. The dual simplex method is carried out on the simplex tableau of the primal problem.

2. Unlike the primal simplex method, we do not require $B^{-1}b$ to be nonnegative. Thus, $x$ needs not be primal feasible.

Question How would you perform the dual simplex algorithm via the revised simplex method?
Example 4.1 Consider the simplex tableau of a minimization problem.

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{c}$</td>
<td>2</td>
<td>6</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$-2$</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$x_5$</td>
<td>4</td>
<td>$-2$</td>
<td>$-3$</td>
<td>0</td>
<td>1</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

1. The given basic solution $\mathbf{x}$ satisfies the optimality conditions but it not feasible. (WHY?)

2. $x_{B(2)} = x_5 < 0$: Choose the $x_5$-row as pivot row.

3. $v_2 = -2 < 0$ and $\bar{c}_2 = 6$: ratio $\bar{c}_2/|v_2| = 3$ (smallest), and
   $v_3 = -3 < 0$ and $\bar{c}_3 = 10$: ratio $\bar{c}_3/|v_2| = 10/3$.
   Thus, the entering variable is $x_2$ and the leaving variable is $x_5$.

Recompute the tableau:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{c}$</td>
<td>14</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>$-3$</td>
</tr>
<tr>
<td>$x_4$</td>
<td>6</td>
<td>0</td>
<td>$-5$</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$-1/2$</td>
<td>1</td>
<td>3/2</td>
<td>0</td>
<td>$-1/2$</td>
<td>1/2</td>
</tr>
</tbody>
</table>

Note: The cost has increased to 3, and the new basic solution is optimal and feasible. An optimal solution is $\mathbf{x} = (0, 1/2, 0, 0, 0)$ with optimal cost 3.

Combine the two tableaus as follows:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{c}$</td>
<td>2</td>
<td>6</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$-2$</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$x_5$</td>
<td>4</td>
<td>$-2$</td>
<td>$-3$</td>
<td>0</td>
<td>1</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\bar{c}$</td>
<td>14</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>$-3$</td>
</tr>
<tr>
<td>$x_4$</td>
<td>6</td>
<td>0</td>
<td>$-5$</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$-1/2$</td>
<td>1</td>
<td>3/2</td>
<td>0</td>
<td>$-1/2$</td>
<td>1/2</td>
</tr>
</tbody>
</table>

When should we use the dual simplex method?

1. A basic solution of the primal problem satisfying optimality conditions is readily available. (Equivalently a basic feasible solution of the dual problem is readily available.)

2. Most importantly, it is used in Sensitivity and Postoptimality analysis. Suppose that we have already an optimal basis for a linear programming problem, and that we wish to solve the same problem for a different choice of vector $\mathbf{b}$. The optimal basis for the original problem may be primal infeasible under the new $\mathbf{b}$. On the other hand, a change in $\mathbf{b}$ does not affect the reduced costs so that optimality conditions are satisfied. Thus, we may apply the dual simplex algorithm starting from the optimal basis for the original problem.
Example 4.2 Solve the LP problem:

Minimize \( 2x_1 + x_2 \)
Subject to \( 3x_1 + x_2 \geq 3 \)
\( 4x_1 + 3x_2 \geq 6 \)
\( x_1 + 2x_2 \leq 3 \)
\( x_1, x_2 \geq 0 \)

Solution
Transform into standard form by multiplying each of the equations associated with the surplus variables \( S_1 \) and \( S_2 \) by \(-1\) so that the RHS will show readily as infeasible basic solution:

Minimize \( 2x_1 + x_2 \)
Subject to \(-3x_1 - x_2 + S_1 = -3\)
\(-4x_1 - 3x_2 + S_2 = -6\)
\( x_1 + 2x_2 + S_3 = 3\)
\( x_1, x_2, S_1, S_2, S_3 \geq 0\)

The associated starting tableau and the subsequent tableaus are:

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{c} )</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( S_1 ) leaves</td>
<td>(-3)</td>
<td>(-1)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>(-3)</td>
</tr>
<tr>
<td>( S_2 ) enters</td>
<td>(-4)</td>
<td>(-3)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>(-6)</td>
</tr>
<tr>
<td>( S_3 )</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( S_2 ) leaves</td>
<td>(-\frac{4}{3})</td>
<td>0</td>
<td>0</td>
<td>(-\frac{1}{3})</td>
<td>0</td>
<td>(-2) ← Always remains optimal ( x_1 ) enters</td>
</tr>
<tr>
<td>( S_1 )</td>
<td>(-\frac{3}{4})</td>
<td>0</td>
<td>1</td>
<td>(-\frac{3}{4})</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( S_3 )</td>
<td>(-\frac{3}{7})</td>
<td>0</td>
<td>0</td>
<td>(-\frac{3}{7})</td>
<td>1</td>
<td>(-1)</td>
</tr>
<tr>
<td>( \bar{c} )</td>
<td>0</td>
<td>0</td>
<td>(-\frac{12}{5})</td>
<td>(-\frac{6}{5})</td>
<td>0</td>
<td>(-\frac{12}{5})</td>
</tr>
<tr>
<td>optimum</td>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>(-\frac{3}{5})</td>
<td>(-\frac{6}{5})</td>
<td>0</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>(-\frac{1}{2})</td>
<td>(-\frac{1}{2})</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( S_3 )</td>
<td>0</td>
<td>0</td>
<td>(-1)</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The graph:
The solution starts at point $A (x_1 = 0, x_2 = 0$ and $S_1 = -3, S_2 = -6, S_3 = 3)$ with cost 0, which is infeasible with respect to the solution space. The next iteration is secured by moving to point $B (x_1 = 0, x_2 = 2)$ with cost 2 which is still infeasible. Finally, we reach point $C (x_1 = \frac{3}{5}, x_2 = \frac{6}{5})$ at which cost $\frac{12}{5}$. This is the first time we encounter a feasible solution, thus signifying the end of the iteration process. Notice that the value of cost associated with $A, B,$ and $C$ are 0, 2, and $\frac{12}{5}$ respectively, which explains why the solution starts at $A$ is better than optimal.

**Note** If instead we let $S_1$ be the leaving variable (forcing the negative basic variable out of the solution), then the iterations would have proceeded in the order $A \rightarrow D \rightarrow C$.

**Example 4.3** Solve by the dual simplex method:

Minimize $2x_1 + 3x_2$

Subject to

$2x_1 + 3x_2 \leq 1$

$x_1 + x_2 = 2$

$x_1, x_2 \geq 0$

**Solution.** We replace the equality constraint by two inequalities to obtain:

Minimize $2x_1 + 3x_2$

Subject to

$2x_1 + 3x_2 + S_1 = 1$

$x_1 + x_2 + S_2 = 2$

$-x_1 - x_2 + S_3 = -2$

$x_1, x_2, S_1, S_2, S_3 \geq 0$

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_3$ leaves</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_1$</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S_2$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$S_3$</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>$\bar{c}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>-4</td>
</tr>
<tr>
<td>$S_1$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>-3</td>
</tr>
<tr>
<td>$S_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

Since $S_1 = -3$, $S_1$ is the leaving variable. However, and all the values in the $S_1$-row are nonnegative. Thus, we conclude that the primal LP is infeasible, i.e. there is no primal feasible solution.
Chapter 6

Sensitivity and Postoptimality Analysis.

In practice, there is often incomplete knowledge of the problem data, and we may wish to predict the effects of certain parameter changes. Sensitivity (or postoptimality) analysis is concerned with the study of possible changes in the available optimal solution as a result of making changes in the original problem.

Consider the standard form problem

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}
\]

or

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}
\]

where \( A \) is \( m \times n \) with linearly independent rows.

We shall study the dependence of the optimal objective value and the optimal solution on the coefficient matrix \( A \), the requirement vector \( b \), and the cost vector \( c \).

Suppose \( x^* \) is an optimal primal solution, with associated optimal basis \( B \). Then \( x_B = B^{-1}b > 0 \) and the optimal cost is \( c_B^T x_B = c_B^T B^{-1}b \).

The optimal basis matrix \( B \) satisfies the following 2 conditions:

\[
\begin{align*}
B^{-1}b \geq 0 & \quad \text{Feasibility} \\
c^T - c_B^T B^{-1}A \geq 0 & \quad \text{Optimality (minimization)}
\end{align*}
\]

OR

\[
\begin{align*}
B^{-1}b \geq 0 & \quad \text{Feasibility} \\
c^T - c_B^T B^{-1}A \leq 0 & \quad \text{Optimality (maximization)}
\end{align*}
\]

Suppose that some entry of \( A \), \( b \) or \( c \) has been changed, or that a new variable is added, or that a new constraint is added. These two conditions
may be affected.

We shall look for conditions under which current basis is still optimal. If the feasibility conditions or optimality conditions are violated, we look for algorithm that finds a new optimal solution without having to solve the new problem from scratch.

### 6.1 A new variable is added.

Consider the standard form problem:

$$
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}
$$

Suppose a new variable $x_{n+1}$, together with a corresponding $A_{n+1}$ and cost $c_{n+1}$ is added. This yields the new problem:

$$
\begin{align*}
\text{minimize} & \quad c^T x + c_{n+1}x_{n+1} \\
\text{subject to} & \quad Ax + A_{n+1}x_{n+1} = b \\
& \quad x \geq 0.
\end{align*}
$$

**Question** Is $B$ still optimal?

First, note that $(x, x_{n+1}) = (x^*, 0)$ is a basic feasible solution to the new problem with basis matrix $B$. Thus, we only need to check whether optimality conditions are satisfied. This amounts to checking whether $\bar{c}_{n+1} = c_{n+1} - c_B^TB^{-1}A_{n+1} \geq 0$ for $B$ to be optimal.

If $\bar{c}_{n+1} \geq 0$, then $(x, x_{n+1}) = (x^*, 0)$ is an optimal solution to the new problem.

If $\bar{c}_{n+1} < 0$, then $(x, x_{n+1}) = (x^*, 0)$ is a basic feasible solution but not necessary optimal. We add a column to the simplex tableau, associated with the new variable, and apply the primal simplex algorithm starting from current basis $B$.

**Remark** If the primal is a maximization problem, then we check whether $\bar{c}_{n+1} = c_{n+1} - c_B^TB^{-1}A_{n+1} \leq 0$ for $B$ to be optimal.
Example 1 Consider the problem

\[
\begin{align*}
\text{minimize} & \quad -5x_1 - x_2 + 12x_3 \\
\text{subject to} & \quad 3x_1 + 2x_2 + x_3 + x_4 = 10 \\
& \quad 5x_1 + 3x_2 + x_4 = 16 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

An optimal solution to this problem is given by \( \mathbf{x} = (2, 2, 0, 0)^T \) and the corresponding optimal simplex tableau is given by

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{c} )</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>2</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>-3</td>
</tr>
</tbody>
</table>

From columns under \( x_3 \) and \( x_4 \), we have

\[
\mathbf{B}^{-1} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}.
\]

Introduce a new variable \( x_5 \) with \( \mathbf{A}_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), and \( \bar{c}_5 = -1 \), we obtain the new problem:

\[
\begin{align*}
\text{minimize} & \quad -5x_1 - x_2 + 12x_3 - x_5 \\
\text{subject to} & \quad 3x_1 + 2x_2 + x_3 + x_5 = 10 \\
& \quad 5x_1 + 3x_2 + x_4 + x_5 = 16 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{align*}
\]

Check \( \bar{c}_5 = c_5 - \mathbf{c}^T \mathbf{B}^{-1} \mathbf{A}_5 \geq 0? \)

\[
\bar{c}_5 = -1 - [-5 - 1] \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -4.
\]

Since \( \bar{c}_5 < 0 \), introducing the new variable to the basis can be beneficial.

Now, \( \mathbf{B}^{-1} \mathbf{A}_5 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \).

We augment the tableau as follows with a new column associated \( x_5 \), and apply primal simplex algorithm:

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{c} )</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>7</td>
<td>-4</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>-3</td>
<td>2</td>
</tr>
<tr>
<td>( \bar{c} )</td>
<td>0</td>
<td>2</td>
<td>12</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0.5</td>
<td>-0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>0</td>
<td>0.5</td>
<td>2.5</td>
<td>-1.5</td>
<td>1</td>
</tr>
</tbody>
</table>

An optimal solution is given by \( \mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T = (3, 0, 0, 0, 1)^T \), with optimal cost \(-16\).
6.2 A new constraint is added.

Consider the standard form problem:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}
\]

Suppose a new constraint \( a_{m+1}^T x \geq b_{m+1} \) is added to the original problem. If the optimal solution \( x^* \) satisfies this new constraint, then the solution remains optimal to the new problem.

If this constraint is violated, introduce a surplus variable \( x_{n+1} \geq 0 \), and rewrite \( a_{m+1}^T x - x_{n+1} = b_{m+1} \).

This yields the new problem:

\[
\begin{align*}
\text{minimize} & \quad c^T x + 0x_{n+1} \\
\text{subject to} & \quad Ax + 0x_{n+1} = b \\
& \quad a_{m+1}^T x - x_{n+1} = b_{m+1} \\
& \quad x = (x_1, x_2, \ldots, x_n)^T \geq 0, \, x_{n+1} \geq 0
\end{align*}
\]

From the optimal basis matrix \( B \) of the original problem, form a new basis \( \tilde{B} \) of the new problem (with an additional basic variable \( x_{n+1} \)). Thus, we have

\[
\tilde{B} = \begin{bmatrix} B & 0 \\ \tilde{a}^T & -1 \end{bmatrix}
\]

where the row vector \( \tilde{a}^T \) contains components of \( a_{m+1}^T \) associated with original basic columns. Then \( \tilde{B} \) is a basis matrix of the new problem (WHY?) and

\[
\tilde{B}^{-1} = \begin{bmatrix} B^{-1} & 0 \\ \tilde{a}^T B^{-1} & -1 \end{bmatrix}. \quad \text{(Check!)}
\]

The corresponding basic solution is \((x^*, a_{m+1}^T x^* - b_{m+1})\) which is infeasible as the new constraint is violated.

The vector of reduced costs associated with \( \tilde{B} \) for the new problem is

\[
\begin{bmatrix} c^T & 0 \\ c_B^T & 0 \end{bmatrix} \begin{bmatrix} B^{-1} & 0 \\ \tilde{a}^T B^{-1} & -1 \end{bmatrix} \begin{bmatrix} A \\ a_{m+1}^T \end{bmatrix} \geq \begin{bmatrix} c^T c_B^T \tilde{B}^{-1} A \\ 0 \end{bmatrix} \geq \begin{bmatrix} 0 & 0 \end{bmatrix}.
\]

Hence, we have obtain an ‘optimal’ but infeasible basic solution to the new problem. Thus, we apply dual simplex method to the new problem, with starting simplex tableau corresponds to the basis \( \tilde{B} \). The reduced cost vector is obtained from \( \begin{bmatrix} c^T c_B^T \tilde{B}^{-1} A \\ 0 \end{bmatrix} \), and constraint coefficients

\[
\tilde{B}^{-1} \begin{bmatrix} A \\ a_{m+1}^T \end{bmatrix} \geq \begin{bmatrix} \tilde{a}^T \tilde{B}^{-1} A \\ a_{m+1}^T \end{bmatrix}.
\]

102
Note that $\mathbf{B}^{-1}\mathbf{A}$ is available from the final simplex tableau for the original problem.

**Exc.** Show that

$$
\mathbf{B}^{-1} \begin{bmatrix}
\mathbf{b} \\
\mathbf{b}_{m+1}
\end{bmatrix} = \begin{bmatrix}
\mathbf{B}^{-1}\mathbf{b} \\
\mathbf{a}^T\mathbf{B}^{-1}\mathbf{b} - \mathbf{b}_{m+1}
\end{bmatrix}.
$$

**Example 2** Consider the same LP problem as in Example 1.

\[
\begin{align*}
\text{minimize} \quad & -5x_1 - x_2 + 12x_3 \\
\text{subject to} \quad & 3x_1 + 2x_2 + x_3 = 10 \\
& 5x_1 + 3x_2 + x_4 = 16 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

with the optimal simplex tableau given by

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{c}$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>2</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>-3</td>
</tr>
</tbody>
</table>

The additional constraint: $x_1 + x_2 \geq 5$ is violated by the optimal solution by $\mathbf{x}^* = (2, 2, 0, 0)^T$.

The new problem is:

\[
\begin{align*}
\text{minimize} \quad & -5x_1 - x_2 + 12x_3 \\
\text{subject to} \quad & 3x_1 + 2x_2 + x_3 = 10 \\
& 5x_1 + 3x_2 + x_4 = 16 \\
& x_1 + x_2 - x_5 = 5 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{align*}
\]

The initial simplex tableau for applying dual simplex method is thus:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{c}$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

where the last row is obtained from:

$$
\mathbf{a}^T\mathbf{B}^{-1}\mathbf{A} - \mathbf{a}_{m+1}^T = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 5 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & -1 \end{bmatrix}.
$$

103
6.3 Changes in the requirement vector $b$.

Suppose that some component $b_i$ of the requirement vector $b$ is changed to $b_i + \delta$, i.e. $b$ is changed to $b + \delta e_i$.

Our aim is to determine the range of values of $\delta$ under which the current basis remains optimal.

Optimality conditions are unaffected by the change in $b$ (WHY?). It remains to examine the feasibility condition

$$
B^{-1}(b + \delta e_i) \geq 0, \quad \text{i.e. } x_B^* - \delta(B^{-1}e_i) \geq 0.
$$

This provides a range for $\delta$ to maintain feasibility (as illustrated in the next example).

However, if $\delta$ is not in the range, then feasibility condition is violated, and we apply dual simplex method starting from the basis $B$.

**Example 3** Consider the same LP problem in Example 1, with the optimal solution $x^* = (2, 2, 0, 0)$ and optimal simplex tableau:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>2</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>-3</td>
</tr>
</tbody>
</table>

(a) Find the range of $b_1$ so that $B$ remains as an optimal basis matrix.

(b) How is the cost affected?

**Solution.**

(a) Suppose $b_1$ is changed to $b_1 + \delta$. Then, the values of basic variables are changed:

$$
x_B = B^{-1}
\begin{bmatrix}
10 + \delta \\
16
\end{bmatrix}
= 
\begin{bmatrix}
-3 & 2 \\
5 & -3
\end{bmatrix}
\begin{bmatrix}
10 + \delta \\
16
\end{bmatrix}
= 
\begin{bmatrix}
2 - 3\delta \\
2 + 5\delta
\end{bmatrix}.
$$

For the new solution to be feasible, both $2 - 3\delta \geq 0$ and $2 + 5\delta \geq 0$, yielding $-2/5 \leq \delta \leq 2/3$.

Thus, the range for $b_1$ is $10 - 2/5 \leq b_1 \leq 10 + 2/3$, i.e. $9\frac{2}{3} \leq b_1 \leq 10\frac{2}{3}$ for $B$ to remain as the basis matrix.

The corresponding change in the cost is $\delta c_B^T B^{-1} e_1 = 10\delta$.

**Note** If $\delta > 2/3$, then $x_1 < 0$ and the basic solution becomes infeasible. We can perform the dual simplex method to remove $x_1$ from the basis and $x_3$ enters the basis.

104
6.4 Changes in the cost vector \( c \).

Consider the standard form problem:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}
\]

Suppose that some component \( c_j \) of the cost vector \( c \) is changed to \( c_j + \delta \).

At an optimal basis matrix \( B \), the primal feasibility condition is not affected. It thus remains to examine the optimality condition

\[
c^T - c_B B^{-1} A \geq 0.
\]

For a nonbasic variable \( x_j \), if \( c_j \) is changed to \( c_j + \delta_j \), then, \( c_B \) is not affected, and only the following inequality is affected

\[
(c_j + \delta_j) - c_B B^{-1} A_j \geq 0,
\]

i.e.

\[
\bar{c}_j + \delta_j \geq 0.
\]

This gives a range for \( \delta_j \), namely, \( \delta_j \geq -\bar{c}_j \).

For a basic variable \( x_j \), if \( c_j \) is changed to \( c_j + \delta_j \), then, \( c_B \) is affected, and hence all the optimality conditions are affected.

We shall illustrated this case in the next example and also determine a range for \( \delta_j \) for a basic variable.

**Example 4** Consider the same LP problem as in Example 1.

\[
\begin{align*}
\text{minimize} & \quad -5x_1 - x_2 + 12x_3 \\
\text{subject to} & \quad 3x_1 + 2x_2 + x_3 = 10 \\
& \quad 5x_1 + 3x_2 + x_4 = 16 \\
& \quad x_1, x_2, x_3, x_4 \geq 0.
\end{align*}
\]

with the optimal solution \( x^* = (2, 2, 0, 0)^T \) and the optimal simplex tableau given by

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>2</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>-3</td>
</tr>
</tbody>
</table>

(a) Determine the range of changes \( \delta_3 \) and \( \delta_4 \) of \( c_3 \) and \( c_4 \) respectively under which the basis remains optimal.

(b) Determine the range of change for \( \delta_1 \) of \( c_1 \) under which the basis remains optimal.
Solution

(a) For nonbasic variables $x_3$ and $x_4$, the corresponding optimality conditions are
\[(c_3 + \delta_3) - c_B^T B^{-1} A_3 \geq 0 \quad \text{and} \quad (c_4 + \delta_4) - c_B^T B^{-1} A_4 \geq 0,\]
i.e.
\[\bar{c}_3 + \delta_3 \geq 0 \quad \text{and} \quad \bar{c}_4 + \delta_4 \geq 0.\]
Therefore, $\delta_3 \geq -\bar{c}_3 = -2$ and $\delta_4 \geq -\bar{c}_4 = -7$.

(b) For basic variables $x_1$ and $x_2$, note that changes in $c_1$ and $c_2$ affect $c_B$. The reduced costs of $x_1$ and $x_2$ are zero. Thus, we need to compute the reduced costs of all nonbasic variables.

The reduced cost of the nonbasic variable $x_3$
\[= c_3 - \left[ c_1 + \delta_1 \ c_2 \right] B^{-1} A_3 \]
\[= (c_3 - c_B^T B^{-1} A_3) - \left[ \delta_1 \ 0 \right] B^{-1} A_3 \]
\[= \bar{c}_3 - \left[ \delta_1 \ 0 \right] \left[ \begin{array}{c} -3 \\ 5 \end{array} \right] = \bar{c}_3 + 3\delta_1 \]
and the reduced cost of the nonbasic variable $x_4$
\[= \bar{c}_4 - 2\delta_1 \ (\text{Check}).\]
Thus, to maintain optimality conditions, we must have
\[\bar{c}_3 + 3\delta_1 \geq 0 \quad \text{and} \quad \bar{c}_4 - 2\delta_1 \geq 0.\]
i.e. $\delta_1 \geq -2/3$ and $\delta_1 \leq 7/2$, i.e. $-2/3 \leq \delta_1 \leq 7/2$. 

106
6.5 Changes in a nonbasic column of A.

At optimal basis $B$. Suppose that some entry $a_{ij}$ of the nonbasic column of $A_j$ is changed to $a_{ij} + \delta$. We wish to determine the range of values of $\delta$ for which the old primal optimal basis matrix remains optimal.

Since $A_j$ is nonbasic, the basis matrix $B$ does not change. Hence, the primal feasibility conditions are unaffected. However, among the reduced costs, only $\bar{c}_j$ is affected. Thus in examining the optimality conditions, we only examine the $j$th-reduced cost:

$$\bar{c}_j = c_j - c^T B^{-1} A_j.$$

If this optimality condition is violated, the old primal optimal solution is feasible but not optimal; and thus we should proceed to apply primal simplex method.

Example 5 Consider the same LP problem as in Example 1

$$\begin{align*}
\text{minimize} & \quad -5x_1 - x_2 + 12x_3 \\
\text{subject to} & \quad 3x_1 + 2x_2 + x_3 = 10 \\
& \quad 5x_1 + 3x_2 + x_4 = 16 \\
& \quad x_1, x_2, x_3, x_4 \geq 0.
\end{align*}$$

Suppose that $A_3$ is changed from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Will the optimal solution $x^* = (2, 2, 0, 0)^T$ be affected?

**Solution** Changing $A_3$ does not affect the optimality condition $B^{-1} b \geq 0$, and the only affected reduced cost is $\bar{c}_3$.

$$\bar{c}_3 = c_3 - c^T B^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 12 - \begin{bmatrix} -5 \\ 5 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 9 \geq 0.$$

Thus, $x^* = (2, 2, 0, 0)^T$ remains as the optimal solution to the new problem.

**NOTE** However, if $A_3$ is to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then the reduced cost $\bar{c}_3 = -1 < 0$. This indicates that $x^* = (2, 2, 0, 0)^T$ a basic feasible solution to the new problem but it is not optimal. Thus, we apply primal simplex method to the following simplex tableau, where the $x_3$-column is replaced by $B^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>7</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>-4</td>
<td>2</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>-3</td>
</tr>
</tbody>
</table>

where $x_3$ enters and $x_2$ leaves the basis.
6.6 Applications.

Example 6.1
Consider the product-mix problem in which each of three products is processed on three operations. The limits on the available time for the three operations are 430, 460, 420 minutes daily and the profits per unit of the three products are $3, $2 and $5. The times in minutes per unit on the three operations are given as follows:

<table>
<thead>
<tr>
<th>Product 1</th>
<th>Product 2</th>
<th>Product 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Operation 1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Operation 2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>Operation 3</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

The LP model is written as:

\[
\text{Maximize} \quad 3x_1 + 2x_2 + 5x_3 \quad \text{(daily profit)}
\]

\[
\text{Subject to} \quad \begin{array}{llll}
3x_1 & + & 2x_2 & + x_3 \leq 430 \quad \text{(unit on usage of op. 1)} \\
3x_1 & + & 2x_3 & \leq 460 \quad \text{(unit on usage of op. 2)} \\
x_1 & + & 4x_2 & \leq 420 \quad \text{(unit on usage of op. 3)}
\end{array}
\]

\[x_1, x_2, x_3 \geq 0.\]

Adding slack variables \(S_1, S_2\) and \(S_3\) to the three constraints, the optimal tableau is given as:

<table>
<thead>
<tr>
<th>Basic</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(S_1)</th>
<th>(S_2)</th>
<th>(S_3)</th>
<th>\text{Solution}</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{c})</td>
<td>-4</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
<td>0</td>
<td>-1350</td>
</tr>
<tr>
<td>(x_2)</td>
<td>-1/4</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>-1/4</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>(x_3)</td>
<td>3/2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>230</td>
</tr>
<tr>
<td>(S_3)</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>20</td>
</tr>
</tbody>
</table>

Which product is not profitable? How to make it profitable?

Solution. Note that the optimal mix does not include Product 1 \((x_1 = 0)\). This means that Product 1 is not profitable, i.e. it is not profitable to produce Product 1.

We wish to make Product 1 profitable, i.e. increase \(x_1\) from 0.

The reduce cost \(\bar{c}_1\) of \(x_1\) is negative \((-4)\). Thus, to make \(x_1\) profitable, we will try to increase the value of \(\bar{c}_1\) to a positive value.

At the current basis \(B\), we have

\[
\bar{c}_1 = c_1 - c_B^T B^{-1} A_1 = c_1 - (a_{11} + 2a_{21} + 0a_{31}),
\]

since \(c_B^T B^{-1} = (p_1, p_2, p_3) = (1, 2, 0)\). Here, \(a_{11}, a_{21}, a_{31}\) are coefficients of \(x_1\) in the functional constraints. These coefficients represent usages by Product 1 of the three operation times.
Thus to make Product 1 more attractive economically (i.e. increase $x_1$ from zero to a positive value), we can either

1. increase the profit (i.e. profit per unit) $c_1$; or

2. decrease the consumption of the limited resources $c_B^T B^{-1} A_1$.

The first option may not always be feasible since profit margins are normally dictated by the market and competition conditions. The second option truly reflects the commitment of economic entity to improving its operation primarily through a reduction in the use of limited resources. In essence, the second option deals with removing possible inefficiencies in the operation of the system under consideration.

Since $c_1 = 3$, the reduced cost $\bar{c}_1 = 3 - a_{11} - 2a_{21}$. Thus, to increase $\bar{c}_1$, we may decrease the value of $a_{11}$ or $a_{21}$. However, since the coefficient of $a_{21}$ is more negative than that of $a_{11}$, it is reasonable to give priority to decrease $a_{21}$ (the usage of Operation 2 by Product 1).

**The immediate question is: How much usage of Operation 2 to reduce?**

Now, we are interested in determining the amount of reduction in the usage of Operation 2 that will make Product 1 just profitable. To do so, let $r_2$ represent reduction in minutes per unit of Product 1 on Operation 2. In this case,

$$\bar{c}_1 = 3 - (a_{11} + 2a_{21}) = 3 - 1 - 2(3 - r_2) = -4 + 2r_2.$$  

Thus, Product 1 becomes just profitable when $\bar{c}_1$ is just above 0, that is, $-4 + 2r_2 > 0$ which yields $r_2 > 2$. This means that the usage of Operation 2 must be reduced by more than 2 minutes to make Product 1 profitable.

Thus, we may consider the change of $a_{21} = 0.5$, while keeping the other usages the same, and proceed to look for the new optimal solution.
Example 6.2
DeChi produces two models of electronic gadgets that use resistors, capacitors and chips. The following table summarizes the data of the situation:

<table>
<thead>
<tr>
<th>Resource</th>
<th>Model 1 (units)</th>
<th>Model 2 (units)</th>
<th>Maximum availability (units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistors</td>
<td>2</td>
<td>3</td>
<td>1200</td>
</tr>
<tr>
<td>Capacitors</td>
<td>2</td>
<td>1</td>
<td>1000</td>
</tr>
<tr>
<td>Chips</td>
<td>0</td>
<td>4</td>
<td>800</td>
</tr>
<tr>
<td>Unit profit ($)</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Let $x_1$ and $x_2$ be the amounts produced of Models 1 and 2, respectively. The following is the corresponding LP problem:

Maximize $3x_1 + 4x_2$
Subject to
$2x_1 + 3x_2 \leq 1200$ (Resistors)
$2x_1 + x_2 \leq 1000$ (Capacitors)
$4x_2 \leq 800$ (Chips)
$x, x_2 \geq 0$

The associated optimal simplex tableau is given as follows:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{c}$</td>
<td>0</td>
<td>0</td>
<td>$-\frac{2}{3}$</td>
<td>$-\frac{1}{4}$</td>
<td>0</td>
<td>$-1750$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>$-\frac{1}{3}$</td>
<td>$\frac{2}{3}$</td>
<td>0</td>
<td>450</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>0</td>
<td>$-2$</td>
<td>2</td>
<td>1</td>
<td>400</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>100</td>
</tr>
</tbody>
</table>

Here, $s_1$, $s_2$ and $s_3$ represent the slacks in the respective constraints.

(a) If the available number of resistors is increased to 1300 units, find the new optimal solution.

(b) If the available number of chips is reduced to 350 units, will you be able to determine the new optimum solution directly from the given information? Explain.

(c) A new contractor is offering DeChi additional resistors at 40 cents each but only if DeChi would purchase at least 500 units. Should DeChi accept the offer?

(d) Find the unit profit range for Model 1 that will maintain the optimality of the current solution.
(e) If the unit profit of model 1 is increased to $6, determine the new solution.

(f) Suppose that the objective function is changed to “maximize $5x_1 + 2x_2$”. Determine the associated optimal solution of the new problem.

**Solution** Optimal Basic variables: $x_1, s_3, x_2$; and $B = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}$.

From the optimal simplex tableau, $B^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & 0 \\ -2 & 2 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$.

(a) If the available number of resistors is increased to 1300 units, i.e. $b_1 = 1300$, the optimality conditions are not affected. We check the feasibility condition, $x_B \geq 0$.

Check: $x_B = B^{-1}b = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & 0 \\ -2 & 2 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1300 \\ 1000 \\ 800 \end{bmatrix} = \begin{bmatrix} 425 \\ 200 \\ 150 \end{bmatrix} \geq 0$.

Thus the basis matrix $B$ is again optimal. The new solution is $x_1 = 450, x_2 = 150$ and the profit is $3x_1 + 4x_2 = 1875$.

(b) If the available number of chips is reduced to 350 units, note that the optimality conditions are unaffected. We check the feasibility condition.

Thus, we reoptimize the problem: note that $c_B^T B^{-1}b = 1750$.

Thus the new optimal solution is $x_1 = 456.25, x_2 = 87.5$ and the profit is $1518.75$. 

111
(c) A new contractor is offering DeChi additional resistors at 40 cents each but only if DeChi would purchase at least 500 units. Thus, we take \( b_1 = 1200 + 500 = 1700 \).

Check the feasibility conditions:

\[
x_B = B^{-1}b = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1700 \\ 1000 \\ 800 \end{bmatrix} = \begin{bmatrix} 325 \\ -600 \\ -350 \end{bmatrix}, \text{ which is not feasible.}
\]

Moreover, \( c^T_B B^{-1}b = 2375 \). So, we reoptimize:

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{c} )</td>
<td>0</td>
<td>0</td>
<td>-\frac{1}{4}</td>
<td>-\frac{1}{4}</td>
<td>0</td>
<td>-2375</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>-\frac{1}{4}</td>
<td>\frac{3}{4}</td>
<td>0</td>
<td>325</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>1</td>
<td>-600</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>\frac{1}{2}</td>
<td>-\frac{1}{2}</td>
<td>0</td>
<td>350</td>
</tr>
<tr>
<td>( \bar{c} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-\frac{3}{8}</td>
<td>-\frac{2}{8}</td>
<td>-2000</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>\frac{1}{2}</td>
<td>-\frac{1}{8}</td>
<td>400</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-\frac{1}{2}</td>
<td>300</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>\frac{1}{4}</td>
<td>200</td>
</tr>
</tbody>
</table>

Thus the new optimal solution is \( x_1 = 400, x_2 = 200 \) and the profit is $2000. Change in profit is $2000 - $1750 = $250; Cost for additional 500 units of resistors is 500 \times 0.4 = $200. Thus, there is a net profit of $50. Hence, DeChi should accept the offer.

(d) To find the unit profit range for Model 1 that will maintain the optimality of the current solution: we want \( \bar{c} \leq 0 \), i.e. \( c^T - c^T_B B^{-1}A \leq 0 \).

From the optimal simplex tableau: \( B^{-1}A = \begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \).

Thus, \( \bar{c}^T = c^T - c^T_B B^{-1}A \)

\[
= (c_1, 4, 0, 0, 0) - (c_1, 0, 4) \begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} = (0, 0, 1, 0, 1) \leq c_1 \leq 8.
\]

This yields \( \frac{8}{3} \leq c_1 \leq 8 \). Thus, the unit profit range for Model 1 that will maintain the optimality of the current solution is between \( \frac{8}{3} \) and 8.
(e) If the unit profit of model 1 is increased to $6, this falls in the range obtained in (d). Thus, the same solution $x_1 = 450, x_2 = 100$ but with profit being $6 \times 450 + 4 \times 100 = 3100$.

(f) Suppose that the objective function is changed to maximize $5x_1 + 2x_2$.

$$\vec{c}^T = (5, 2, 0, 0, 0) - (5, 0, 2) \begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} = (0, 0, -\frac{11}{4}, 0),$$

which is nonoptimal.

$$\vec{c}^T B^{-1} b = 5 \times 450 + 2 \times 100 = 2450.$$  
Reoptimize by primal simplex algorithm:

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{c}$</td>
<td>0</td>
<td>0</td>
<td>$-\frac{1}{4}$</td>
<td>$-\frac{11}{4}$</td>
<td>0</td>
<td>$-2450$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>$-\frac{1}{4}$</td>
<td>$\frac{3}{4}$</td>
<td>0</td>
<td>450</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>1</td>
<td>400</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>$\bar{c}$</td>
<td>0</td>
<td>$-\frac{1}{4}$</td>
<td>0</td>
<td>$-\frac{11}{4}$</td>
<td>0</td>
<td>$-2500$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>$\frac{1}{4}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>500</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>800</td>
</tr>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>$-1$</td>
<td>0</td>
<td>200</td>
</tr>
</tbody>
</table>

Optimal solution: $x_1 = 50, x_2 = 0$ & Profit $2500$.  

113
Chapter 7

Transportation Problems.

7.1 Transportation Models and Tableaus.

The transportation model deals with determining a minimum cost plan for transporting a single commodity from a number of sources (such as factories) to a number of destinations (such as warehouses).

Basically, the model is a linear program that can be solved by the regular simplex method. However, its special structure allows the development of a solution procedure, called the transportation algorithm, that is computationally more efficient.

The transportation model can be depicted as a network with $m$ sources and $n$ destinations as follows:

<table>
<thead>
<tr>
<th>Units of supply</th>
<th>Sources</th>
<th>Destinations</th>
<th>Units of demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1</td>
<td>1</td>
<td>$b_1$</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>$a_i$</td>
<td>$i$</td>
<td>$j$</td>
<td>$b_j$</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>$a_m$</td>
<td>$m$</td>
<td>$n$</td>
<td>$b_n$</td>
</tr>
</tbody>
</table>

where $c_{ij}$ is the unit transportation cost between source $i$ and destination $j$.

The objective of the model is to determine $x_{ij}$ which is the amount to be transported from source $i$ to destination $j$ so that the total transportation cost is minimum.
It can be represented by the following LP:

Minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ (cost)

Subject to

$\sum_{j=1}^{n} x_{ij} \leq a_i, i = 1, 2, \cdots m$

(sum of shipments from source $i$ cannot exceed its supply)

$\sum_{i=1}^{m} x_{ij} \geq b_j, j = 1, 2, \cdots n$

(sum of shipments to destination $j$ must satisfy its demand)

$x_{ij} \geq 0, i = 1, 2, \cdots m, j = 1, 2, \cdots n$

The first two sets of constraints imply

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \leq \sum_{i=1}^{m} a_i \Rightarrow \sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$$

i.e. total supply must be at least equal to total demand.

When $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_i$, the resulting formulation is called a balanced transportation model. It differs from the model above only in the fact all constraints are equations, that is

$$\sum_{j=1}^{n} x_{ij} = a_i, i = 1, 2, \cdots m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, j = 1, 2, \cdots n$$

The transportation algorithm to be introduced works on a balanced transportation model.

When the transportation problem is not balanced, i.e. $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_i$, we can balance it by adding dummy source or a dummy destination. We shall discuss unbalanced problems in the last section.

Transportation problem as an LP problem.

Example 1.1

G Auto has three plants in Los Angeles, Detroit, and New Orleans, and two major distribution centers in Denver and Miami. The capacities of the three plants during the next quarter are 1,000, 1,500 and 1,200 cars. The quarterly demand at the two distribution centers are 2,300 and 1,400 cars.

The transportation cost per car on the different routes, rounded to the nearest dollar, are calculated as given in Table 1-1.

<table>
<thead>
<tr>
<th>Plant</th>
<th>Denver</th>
<th>Miami</th>
</tr>
</thead>
<tbody>
<tr>
<td>Los Angeles</td>
<td>$80</td>
<td>$215</td>
</tr>
<tr>
<td>Detroit</td>
<td>$100</td>
<td>$108</td>
</tr>
<tr>
<td>New Orleans</td>
<td>$102</td>
<td>$68</td>
</tr>
</tbody>
</table>

115
Represent the transportation problem as an LP problem.

**Solution**

The LP model of the problem in Table 1-1:

Minimize \( 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32} \)

Subject to

\[
\begin{align*}
  x_{11} + x_{12} &= 1000 \\
  x_{21} + x_{22} &= 1500 \\
  x_{31} + x_{32} &= 1200 \\
  x_{11} + x_{21} + x_{31} &= 2300 \\
  x_{12} + x_{22} + x_{32} &= 1400 \\
  x_{ij} &\geq 0, \ i = 1, 2, 3; \ j = 1, 2.
\end{align*}
\]

Note that these constraints are equations because the total supply from the three sources equals the total demand at the two destinations. This is a balanced transportation model.

**Transportation tableau**

The transportation tableau is used instead of the simplex tableau as illustrated in the following example.

**Example 1.2** The transportation tableau of Example 1.1:

<table>
<thead>
<tr>
<th></th>
<th>Denver</th>
<th>Miami</th>
</tr>
</thead>
<tbody>
<tr>
<td>Los Angeles</td>
<td>80</td>
<td>215</td>
</tr>
<tr>
<td></td>
<td>(x_{11})</td>
<td>(x_{12})</td>
</tr>
<tr>
<td>Detroit</td>
<td>100</td>
<td>108</td>
</tr>
<tr>
<td></td>
<td>(x_{21})</td>
<td>(x_{22})</td>
</tr>
<tr>
<td>New Orleans</td>
<td>102</td>
<td>68</td>
</tr>
<tr>
<td>Demand</td>
<td>2300</td>
<td>1400</td>
</tr>
<tr>
<td></td>
<td>(x_{31})</td>
<td>(x_{32})</td>
</tr>
</tbody>
</table>

**Remark** In the transportation tableau, the \((i, j)\)-cell in the \(i\)-row and \(j\)-column represents the decision variable \(x_{ij}\). We write the unit transportation cost from source \(i\) to destination \(j\) on the top right hand corner of the \((i, j)\)-cell.
Number of basic variables

**Proposition 7.1.1** The balanced transportation problem has $m + n - 1$ basic variables.

**Proof:** The number of basic variables equals to the number of linearly independent equality constraints.

The coefficient matrix of equality constraints is represented as follows:

$$
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \ddots & \ddots & \ddots \\
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\end{pmatrix}
$$

The sum of first $m$ rows minus the sum of last $n$ rows equals to 0. Thus, the rank of the matrix $\leq m + n - 1$.

On the other hand, we can find $m + n - 1$ linearly independent columns, e.g. \{ first $n$ columns, $(n+1)$-column, $(2n+1)$-column, \ldots, $[(m-1)n-1]$-column \}. Therefore, the rank is $m + n - 1$. 

QED
7.2 The Transportation Algorithm

The transportation algorithm works on a balanced transportation model. The steps of the transportation algorithm are exact parallels of the simplex method, namely:

Step 1 Determine a starting basic feasible solution, and go to Step 2.

Step 2 Use the optimality condition of the simplex method to determine the entering variable from among the nonbasic variables. If the optimality condition is satisfied, stop. Otherwise, go to Step 3.

Step 3 Use the feasibility condition of the simplex method to determine the leaving variable from among all the current basic variables, and find the new basic variable. Return to Step 2.

However, we take advantage of the special structure of the transportation model to present the algorithm in a more convenient form. Each of the above steps is detailed subsequently.

Example 2.1: The Sun Ray Company ships truckloads of grain from three silos to four mills. The supply (in truckloads) and the demand (also in truckloads) together with the unit transportation costs per truckload on the different routes are summarized in the transportation model in Table 2-1. The unit transportation costs, \( c_{ij} \), (shown in the northeast corner of each box) are in hundreds dollars.

Table 2-1

<table>
<thead>
<tr>
<th></th>
<th>Mill 1</th>
<th>Mill 2</th>
<th>Mill 3</th>
<th>Mill 4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silo 1</td>
<td>x_{11}</td>
<td>x_{12}</td>
<td>x_{13}</td>
<td>x_{14}</td>
<td>15</td>
</tr>
<tr>
<td>Silo 2</td>
<td>x_{21}</td>
<td>x_{22}</td>
<td>x_{23}</td>
<td>x_{24}</td>
<td>20</td>
</tr>
<tr>
<td>Silo 3</td>
<td>x_{31}</td>
<td>x_{32}</td>
<td>x_{33}</td>
<td>x_{34}</td>
<td>25</td>
</tr>
<tr>
<td>Demand</td>
<td>5</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>10</td>
</tr>
</tbody>
</table>

The purpose of the model is to determine the minimum cost shipping schedule between the silos and the mills, i.e. to determining the quantity \( x_{ij} \) shipped from silo \( i \) to mill \( j \) \((i = 1, 2, 3; j = 1, 2, 3, 4)\).

**Step 1.** Determine a starting basic feasible solution.

For a general transportation tableau of size \( m \times n \), there are \( m + n - 1 \) basic variables. Three different procedures will be discussed: (1) **Northwest-corner Method**, (2) **Least-cost Method**, and (3) **Vogel’s Approximation Method** (VAM).
1. **Northwest-corner Method** starts at the northwest corner cell \( x_{11} \) of the tableau.

Step 1 Allocate as much as possible to the selected cell, and adjust the associated amount of supply and demand by subtracting the allocated amount.

Step 2 Cross out the row or column with zero supply or demand to indicate that no further assignments can be made in that row or column. If both the column and row net to zero simultaneously, cross out **one only** (either one), and leave a zero supply (demand) in the uncrossed-out row (column).

Step 3 Move to the cell to the right if a column has just been crossed or the one below if a row has been crossed out. Go to Step 1.

**Example 2-1. NW corner method**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>2</td>
<td>20</td>
<td>11</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>7</td>
<td>9</td>
<td>20</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>10</td>
</tr>
<tr>
<td>Demand</td>
<td>5</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>40</td>
</tr>
</tbody>
</table>

The basic variables of the starting basic solution is

\[
x_{11} = 5 \quad x_{12} = 10 \\
x_{22} = 5 \quad x_{23} = 15 \quad x_{24} = 5 \\
x_{34} = 10
\]

Total cost = 5(10) + 10(2) + 5(7) + 15(9) + 5(20) + 10(18) = 520.

**Note** There are 3 + 4 - 1 = 6 basic variables in the starting basic feasible solution.
2. **Least-cost Method** finds a better starting solution by concentrating on the cheapest routes. It starts at the cell with the smallest unit cost.

**Step 1** Assign as much as possible to the variable with the smallest unit cost in the entire tableau. (Ties are broken arbitrarily.) Adjust the associated amount of supply and demand by subtracting the allocated amount.

**Step 2** Cross out the satisfied row or column. As in the northwest-corner method, if a column and a row are satisfied simultaneously, cross out **one only**.

**Step 3** Move to the uncrossed-out cell with the smallest unit cost. Go to **Step 1**.

**Example 2-2. Least-cost method**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>2</td>
<td>20</td>
<td>11</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>7</td>
<td>9</td>
<td>20</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>10</td>
</tr>
</tbody>
</table>

Source | Demand | 5 | 15 | 15 | 15 |

The basic variables of the starting basic feasible solution is

\[
x_{12} = 15 \quad x_{14} = 0 \\
x_{23} = 15 \quad x_{24} = 10 \\
x_{31} = 5 \quad x_{34} = 5
\]

and the associated cost is

\[
15(2) + 0(11) + 15(9) + 10(20) + 5(4) + 5(18) = 475.
\]
3. **Vogel’s Approximation Method (VAM)** is an improved version of the least-cost method that generally produces better starting solutions.

**Step 1** For each row (column) with strictly positive supply (demand), evaluate a penalty measure by subtracting the \textit{smallest} cost element in the row (column) from the \textit{next smallest} cost element in the same row (column).

**Step 2** Identify the row (column) with the largest penalty, breaking ties arbitrarily. Allocate as much as possible to the variable with the least unit cost in the selected row (column). Adjust the supply and demand and cross out the satisfied row (column). If a column and a row are satisfied simultaneously, crossed out the row (column) with the largest penalty and the remaining column (row) is assigned a zero demand (supply).

**Step 3** Recompute the penalties for the uncrossed out rows and columns, then go to Step 2.

**Remark**

1. The row and column penalties are the penalties that will be incurred if, instead of shipping over the \textit{best} route, we are forced to ship over the \textit{second-best} route. The most serious one (largest penalty) is selected and allocate as much as possible to the variable with the smallest unit cost.

2. The variable at the selected cell must be regarded as a basic variable even if it is assigned zero amount.
Example 2-3. Vogel’s method

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>2</td>
<td>20</td>
<td>11</td>
<td>15</td>
</tr>
<tr>
<td>Source 2</td>
<td>12</td>
<td>7</td>
<td>9</td>
<td>20</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>10</td>
</tr>
<tr>
<td>Demand</td>
<td>5</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

The basic variables of the starting basic feasible solution is

\[
\begin{align*}
x_{12} &= 15 \\
x_{22} &= 0 \\
x_{23} &= 15 \\
x_{24} &= 10 \\
x_{31} &= 5 \\
x_{34} &= 5
\end{align*}
\]

and the associated cost is

\[
15(2) + 0(11) + 15(9) + 10(20) + 5(4) + 5(18) = 475.
\]

Same as the solution obtained by the least-cost method.
An example for comparing the three methods.

NW-Corner Method:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source 1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>Source 2</td>
<td>12</td>
<td>20</td>
<td>8</td>
<td>10</td>
<td>25</td>
</tr>
<tr>
<td>Source 3</td>
<td>6</td>
<td>30</td>
<td>9</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>Demand</td>
<td>25</td>
<td>15</td>
<td>10</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

The basic variables of the starting basic feasible solution is

and the associated cost is

Least-Cost Method:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source 1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>Source 2</td>
<td>12</td>
<td>20</td>
<td>8</td>
<td>10</td>
<td>25</td>
</tr>
<tr>
<td>Source 3</td>
<td>6</td>
<td>30</td>
<td>9</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>Demand</td>
<td>25</td>
<td>15</td>
<td>10</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

The basic variables of the starting basic feasible solution is
and the associated cost is

Vogel’s Approximation Method:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>20</td>
<td>8</td>
<td>10</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>30</td>
<td>9</td>
<td>20</td>
<td>15</td>
</tr>
</tbody>
</table>

The basic variables of the starting basic feasible solution is

and the associated cost is
Step 2 Determine an entering variable.

After determining a basic feasible solution, we use the Method of Multipliers (or UV method) to compute the reduced costs of nonbasic variables $x_{pq}$. If the optimality conditions are satisfied, the basic feasible solution is optimal. Otherwise, we proceed to determine the entering variable among the current nonbasic variables (those not part of the starting basic solution).

Method of Multipliers.
Primal:

Minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$

Subject to

\[
\begin{align*}
x_{11} + x_{12} + \cdots + x_{1n} &= a_1 \\
x_{21} + x_{22} + \cdots + x_{2n} &= a_2 \\
\vdots &\quad\vdots \\
x_{m1} + x_{m2} + \cdots + x_{mn} &= a_m \\
x_{11} + x_{21} + \cdots + x_{m1} &= b_1 \\
x_{12} + x_{22} + \cdots + x_{m2} &= b_2 \\
\vdots &\quad\vdots \\
x_{1n} + x_{2n} + \cdots + x_{mn} &= b_n \\
x_{ij} &\geq 0, \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n
\end{align*}
\]

Dual variables

\[
\begin{align*}
u_1 \\
u_2 \\
\vdots \\
u_m \\
v_1 \\
v_2 \\
\vdots \\
v_n
\end{align*}
\]

Dual:

Maximize $\sum_{i=1}^{m} a_iu_i + \sum_{j=1}^{n} b_jv_j$

Subject to

$u_i + v_j \leq c_{ij}, \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$

$u_i, v_j$ unrestricted in sign

NOTES

1. At a basic feasible solution, with basis $B$, we let

$\mathbf{p}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$.

Thus, the reduced cost of $x_{ij}$ is

$\bar{c}_{ij} = c_{ij} - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_{ij} = c_{ij} - \mathbf{A}_{ij}^T \mathbf{p}$.

2. The reduced cost of $\bar{c}_{ij}$ of a basic variable $x_{ij}$ must be zero. Thus, we have

$u_i + v_j = c_{ij}$ for each basic variable $x_{ij}$

These give $m+n-1$ equations in $m+n$ variables $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$. Thus, we set $u_1 = 0$, and use the equations to solve for the remaining variables $u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$. 

125
3. The reduced cost of a nonbasic variable $x_{pq}$ can be computed as follows:

$$
\tau_{pq} = c_{pq} - (u_p + v_q).
$$

4. Since the transportation problem is a minimization problem, the entering variable is a nonbasic variable with negative $\tau_{pq}$.

The UV method involves computation of reduced costs of nonbasic variables via the introduction of multipliers (which are dual variables) $u_i$ and $v_j$. However, the special structure of the transportation model allows simpler computations.

**Summary of steps to determine an entering variable.**

1. Associate the multipliers $u_i$ and $v_j$ with row $i$ and column $j$ of the transportation tableau.

2. For each basic variable $x_{ij}$, solve for values of $u_i$ and $v_j$ from the following equations:

$$
u_i + v_j = c_{ij}
$$

by arbitrarily setting $u_1 = 0$.

3. For each nonbasic variable $x_{pq}$, compute $\tau_{pq} = c_{pq} - (u_p + v_q)$. If $\tau_{pq} \geq 0$ for all nonbasic $x_{pq}$, stop and conclude that the starting feasible solution is optimal.

Otherwise, choose $x_{pq}$ corresponding to a negative value $\tau_{pq}$ to be the entering variable.

**Example 2-4** We use the starting basic feasible solution in Example 2-1, which is obtained by North-west Corner Method:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>10(5)</td>
<td>2(10)</td>
<td>20(10)</td>
<td>11</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>12(5)</td>
<td>7(15)</td>
<td>9</td>
<td>20(5)</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>14</td>
<td>16</td>
<td>18(10)</td>
<td>10</td>
</tr>
<tr>
<td>Demand</td>
<td>5</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>
**Step 3** Determine the leaving variable.

The leaving variable is chosen from the current basic variables (part of the starting basic solution) by the following steps.

1. Construct a closed loop that starts and ends at the entering variable. The loop consists of connected horizontal and vertical segments (no diagonals are allowed). Each corner of the resulting loop, with the exception of that in the entering variable cell, must coincide with a current basic variable.

   (Exactly one loop exists for a given entering variable.)

2. Assign the amount $\theta$ to the entering variable cell. Alternate between subtracting and adding the amount $\theta$ at the successive corners of the loop.

   (In the tableau, starting with $(-)$, indicate signs $(-)$ or $(+)$ alternatively in the south corner of each cell corresponds to a current basic variable at corners.)

3. Choose the largest possible value of $\theta > 0$ such that for each current basic variable $x_{ij}$, we have $x_{ij} \pm \theta \geq 0$ (according to the sign assigned in Step 2). Choose the basic variable $x_{ij}$ corresponding to yielding this largest allowable value of $\theta$ as the leaving variable.

   (In the tableau, the leaving variable is selected among the corner basic variables of the loop labeled $(-)$ and has the smallest value $x_{ij}$.)

**The next basic feasible solution.**

The value of the entering variable $x_{pq}$ is increased to $\theta$, the maximum value found in Step 3. Each value of the corner (basic) variables is adjusted accordingly to satisfy the supply (demand). The new solution is thus obtained. **The new cost.**

The transportation cost of each unit transported through the new route via the entering variable $x_{pq}$ is changed by $\bar{c}_{pq} = c_{pq} - (u_p + v_q)$. Thus the total reduced transportation cost associated with the new route is $\theta \bar{c}_{pq}$. 
**Example 2-5** In Example 2-4, we have found that the entering variable is $x_{31}$. Based on the same starting basic feasible solution, we form a close loop $x_{31} \rightarrow x_{11} \rightarrow x_{12} \rightarrow x_{22} \rightarrow x_{24} \rightarrow x_{34} \rightarrow x_{31}$.

We assign a value $\theta$ to $x_{31}$, and alternate the signs of $\theta$ along the loop.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>(10)</td>
<td>20</td>
<td>11</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>7</td>
<td>9</td>
<td>(5)</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>$\theta$</td>
<td>14</td>
<td>16</td>
<td>(10)</td>
<td>10</td>
</tr>
<tr>
<td>Demand</td>
<td>5</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>2</td>
<td>20</td>
<td>11</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>7</td>
<td>9</td>
<td>20</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>10</td>
</tr>
<tr>
<td>Demand</td>
<td>5</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>2</td>
<td>20</td>
<td>11</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>7</td>
<td>9</td>
<td>20</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>10</td>
</tr>
<tr>
<td>Demand</td>
<td>5</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>
7.3 Unbalanced Transportation model.

The transportation algorithm works on a balanced transportation model. If the given model is not balanced, we will balance it before we carry out the transportation algorithm. A transportation model can always be **balanced** by introducing a **dummy source** or **dummy destination** as follows:

1. If \( \sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_i \), a dummy destination is used to **absorb the surplus**
   \[ \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_i \]
   with unit transportation cost equal to zero or stated storage costs at the various sources.

2. If \( \sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_i \), a dummy source is used to **supply the shortage**
   amount by \( \sum_{j=1}^{n} b_i - \sum_{i=1}^{m} a_i \) with unit transportation cost equal to zero or stated penalty costs at the various destinations for unsatisfied demands.

**Example 3.1**
Telly’s Toy Company produces three kinds of dolls: the Bertha doll, the Holly doll, and the Shari doll in quantities of 1,000, 2,000 and 2,000 per week respectively. These dolls are demanded at three large department stores: Shears, Nicholas and Words. Contract requiring 1,500 total dolls per week are to be shipped to each store. However, Words does not want any Bertha dolls. Because of past contract commitments and size of other orders, profile vary from store to store on each kind of doll. A summary of the unit profit per doll is given below:

<table>
<thead>
<tr>
<th></th>
<th>Shears</th>
<th>Nicholas</th>
<th>Words</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bertha</td>
<td>5</td>
<td>4</td>
<td>-</td>
</tr>
<tr>
<td>Holly</td>
<td>16</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>Shari</td>
<td>12</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

(a) Set up the problem as a transportation problem.

(b) Obtain a starting basic feasible solution by the VAM and proceed to find an optimal solution.

(c) Obtain an alternative optimal solution.
Solution (a) The objective is to maximize the profit which can be converted to a minimization problem with the transportation cost being the negative of the profit.

\[
\begin{array}{cccc}
\text{Shears} & \text{Nicholas} & \text{Words} & \text{Supply} \\
\hline
\text{Bertha} & -5 & -4 & M & 1000 \\
\text{Holly} & -16 & -8 & -9 & 2000 \\
\text{Shari} & -12 & -10 & -11 & 2000 \\
\text{Demand} & 1500 & 1500 & 1500 \\
\end{array}
\]

We have assigned a value of \(+M\) to the cell from Bertha to Words as ‘Words does not want any Bertha dolls’. This large unit transportation cost ensures that the corresponding variable assumes zero value.

Remark In general, unacceptable transportation routes would be assigned a unit transportation cost value of \(+M\).

(b) The transportation problem is not balanced. Thus, we introduce a dummy store to form a balanced transportation model. The unit transportation cost at each dummy cell is assigned to be 0.
Using the UV-method iteratively:

\[
\begin{array}{cccc|c}
& S & N & W & \text{Dummy} \\
B & -5 & -4 & M & 0 \\
H & -16 & -8 & -9 & 0 \\
S & -12 & -10 & -11 & 0 \\
\end{array}
\]

Supply: 1000, 2000, 2000

Demand: 1500, 1500, 1500, 500

Therefore the optimal solution:

<table>
<thead>
<tr>
<th>Doll</th>
<th>Store</th>
<th>Number</th>
<th>Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bertha</td>
<td>Nicholas</td>
<td>500</td>
<td>2,000</td>
</tr>
<tr>
<td>Holly</td>
<td>Shears</td>
<td>1,500</td>
<td>24,000</td>
</tr>
<tr>
<td>Holly</td>
<td>Words</td>
<td>500</td>
<td>4,500</td>
</tr>
<tr>
<td>Shari</td>
<td>Nicholas</td>
<td>1,000</td>
<td>10,000</td>
</tr>
<tr>
<td>Shari</td>
<td>Words</td>
<td>1,000</td>
<td>11,000</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td><strong>51,000</strong></td>
</tr>
</tbody>
</table>
(c) An alternative optimal solution:

![Optimal solution table]

<table>
<thead>
<tr>
<th>Demand</th>
<th>S</th>
<th>N</th>
<th>W</th>
<th>Dummy</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>1500</td>
<td>1500</td>
<td>1500</td>
<td></td>
<td>1000</td>
</tr>
<tr>
<td>H</td>
<td>1500</td>
<td>1500</td>
<td>1500</td>
<td></td>
<td>2000</td>
</tr>
<tr>
<td>S</td>
<td>1500</td>
<td>1500</td>
<td>1500</td>
<td></td>
<td>2000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Doll</th>
<th>Store</th>
<th>Number</th>
<th>Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bertha</td>
<td>Nicholas</td>
<td>500</td>
<td>2,000</td>
</tr>
<tr>
<td>Holly</td>
<td>Shears</td>
<td>1,500</td>
<td>24,000</td>
</tr>
<tr>
<td>Holly</td>
<td>Nicholas</td>
<td>500</td>
<td>4,000</td>
</tr>
<tr>
<td>Shari</td>
<td>Nicholas</td>
<td>500</td>
<td>5,000</td>
</tr>
<tr>
<td>Shari</td>
<td>Words</td>
<td>1,500</td>
<td>16,500</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td><strong>51,000</strong></td>
<td></td>
</tr>
</tbody>
</table>