Q1. Let \( P = \{ x \in \mathbb{R}^n \mid Ax = b, x \geq 0 \} \). Then, there exist \( x^1, \ldots, x^q, d^1, \ldots, d^r \) in \( \mathbb{R}^n \) such that

\[
P = \text{conv}\{x^1, \ldots, x^q\} + \text{cone}\{d^1, \ldots, d^r\}.
\]

Consider the following linear programming problem

\[
\begin{array}{ll}
\min & c^T x \\
\text{s.t.} & x \in P.
\end{array}
\] (1)

(i) The optimal value in (1) is bounded if and only if

\[
c^T d^i \geq 0, \ i = 1, \ldots, r.
\]

(ii) Suppose that the optimal value in (1) is bounded and \( P \) has at least one extreme point. Show that to solve (1) is equivalent to solve

\[
\begin{array}{ll}
\min & \{c^T x^i \mid i = 1, \ldots, q\}.
\end{array}
\] (2)

in the sense that their optimal values are equal and, if the optimal solution of (1) is unique, then the two problems have the same optimal solution.

Q2. Consider a nonempty polyhedron \( P \) and suppose that for each variable \( x_i \) we have either the constraint \( x_i \geq 0 \) or the constraint \( x_i \leq 0 \). Is it true that \( P \) has at least one basic feasible solution?

Q3. Let \( P \) be a bounded polyhedron in \( \mathbb{R}^n \), let \( a \) be a vector in \( \mathbb{R}^n \), and let \( b \) be some scalar. We define

\[
Q = \{ x \in P \mid a^T x = b \}.
\]

Show that every extreme point of \( Q \) is either an extreme point of \( P \) or a convex combination of two adjacent extreme points of \( P \). (Two distinct basic solutions to a set of linear constraints in \( \mathbb{R}^n \) are said to be adjacent if we can find \( n-1 \) linearly independent constraints that are active at both of them.)

Q4. Let \( P \) and \( Q \) be polyhedra in \( \mathbb{R}^n \). Let \( P + Q = \{ x + y \mid x \in P, y \in Q \} \).

(a) Show that \( P + Q \) is a polyhedron.

(b) Show that every extreme point of \( P + Q \) is the sum of an extreme point of \( P \) and an extreme point of \( Q \).