Q1. Write down formulae for computing $\bar{\theta}$ in Algorithm 4.3 Step 3.

Q2. Let $({\Delta x, \Delta \pi, \Delta s})$ be Newton direction generated by (4.28) at a point $(x, \pi, s) \in N^{-\infty}(\eta)$ with $\eta \in (0, 1)$. Suppose that we have proved $\|X^s\|_\infty \leq \eta \mu$, where $\mu = x^T s/n$.

(i) Let $\tilde{\theta}$ be the maximum $\theta$ such that $(x(\theta), \pi(\theta), s(\theta)) \in N^{-\infty}(\eta)$. Show that $\tilde{\theta} \geq \gamma \eta / n$.

(ii) Estimate the number of iterations needed by Algorithm 4.3 to reduce $\mu_k$ to less than $\epsilon$ by using the result obtained in (i).

Q3. Let $({\Delta x, \Delta \pi, \Delta s})$ be the Newton direction at a point $(x, \pi, s) \in N_2(\eta)$ for some $\eta \in (0, 1)$, defined by

$$A\Delta x = 0, \ A^T \Delta \pi + \Delta s = 0, \ X\Delta s + S\Delta x = \gamma \mu e - Xs$$

for some $\gamma \in (0, 1)$, and

$$(x(\theta), \pi(\theta), s(\theta)) = (x, \pi, s) + \theta(\Delta x, \Delta \pi, \Delta s).$$

(i) Prove

$$\|X^s\|_2 \leq \frac{\sqrt{2}(1 - \gamma) \sqrt{n} + \eta^2}{4(1 - \gamma)}.$$  

(ii) Show that $(x(\theta), \pi(\theta), s(\theta)) \in N_2(\eta)$ for all $\theta \in [0, 1]$ if

$$\gamma = 1 - \frac{\eta}{\sqrt{n}}$$

and

$$\frac{(1 - \eta)^2}{\sqrt{2} \eta} \geq 1.$$  

(iii) Consider Algorithm 4.3, but with the small neighborhood $N_2(\eta)$ and $\gamma = 1 - \frac{\eta}{\sqrt{n}}$ for some $\eta \in (0, 1)$ satisfying (2). Show that the algorithm can find a $\mu^k$ less than $\epsilon$ ($> 0$) in at most $O(\sqrt{n} \ln(\mu^0 / \epsilon))$ iterations. Can you prove a better complexity bound (lower than $O(\sqrt{n} \ln(\mu^0 / \epsilon))$) for such an algorithm?