A Study of Piecewise Smooth Demand Functions and Pricing Models

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Abstract

A good model of the relationship between the market demand and prices plays a fundamental role in much economic decision-making. However, many decision-making models use either incomplete demand functions which are defined only on a restricted domain, or functions that do not reflect market reality. In the first part of this paper, we explain our motivation behind a complete reasonable definition of demand functions using some examples. Indeed, these examples show that incomplete demand functions may lead to inferior pricing models. Then we formulate the demand functions using a Nonlinear Complementarity Problem (NCP). We will show that such demand functions possess certain desirable properties, such as the law of demand. In the second part of the paper, we consider an oligopolistic market, where producers/sellers are playing a non-cooperative game to determine prices of their products. When the model of demand functions is incorporated into the best response problem of each producer/seller involved, it leads to a complementarity constrained pricing problem facing each producer/seller. Some basic properties of the pricing models are presented. In particular, we show that, under certain conditions, the complementarity constraints in this pricing model can be eliminated, which tremendously simplifies the computation and theoretical analysis.

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1 Introduction


Linear demand functions are most frequently used in pricing models, see for example, Bernstein and Federgruen (2003), Bernstein and Federgruen (2004), Dai, Chao, Fang and Nuttle (2004), Eliashberg and Steinberg (1991), Federgruen and Meissner (2004), Garcia-Gallego and Georgantzis (2001), Giraud-Heraud, Hammoudi and Mokrane (2003), and Maglaras and Meissner (2003). However, we have not seen any pricing model which explicitly defines the associated demand function on the entire set of nonnegative prices. The difficulty lies in that a linear function usually cannot be nonnegative for all prices. Thus, it requires a proper reconfiguration to ensure nonnegativity of a demand function, and the resulting pricing models will become much more involved.

Let us be more specific. A commonly used linear demand function is of the form $d(p) = b - Ap$, where $d$ and $p$ are the demand and price vectors respectively, $b$ is a constant vector and $A$ is a matrix of appropriate dimensions. By the law of demand, i.e., the demand $d_i$ decreases in its own price $p_i$, the demand function $d_i$ will be negative for a large value of $p_i$. Thus, the linear demand function $d$ is meaningful (nonnegative) only on $\Omega$, defined as the set of nonnegative prices at which all components of $b - Ap$ are nonnegative. It is not customary to consider a demand function of multiple products which evaluates the demand at any $p \notin \Omega$. In the above papers, this problem is evaded by defining demand functions only for prices in $\Omega$, explicitly or implicitly. All $p \notin \Omega$ are deemed infeasible in the pricing models.

But why must all feasible prices be confined to $\Omega$? Are all $p \notin \Omega$ really redundant? The possible desirability of extending the domain of the demand function was briefly discussed by Shubik and Levitan (1980). We wish to argue in the next section that such an extension is not merely desirable: it is necessary in the sense that, in some applications, failing to do so will lead to erroneous conclusions.

Now a piecewise linear demand function for a single product is easy to construct, namely, $D(p) = b - Ap$ for $p \in \Omega$ and $D(p) = 0$ for $p \notin \Omega$. However, the construction of a demand function for multiple products is no longer trivial, because all demands at prices outside $\Omega$ need not vanish. Shubik and Levitan (1980) proposed a scheme to extend any given function to the entire set of nonnegative prices. In this paper, we define a demand function using a nonlinear complementarity problem (in short, NCP). The two formulations essentially share the same spirit.

The main contributions of the first part of this paper are: (i) We explain the validity of
this demand function for a general market: products can be either substitutes or comple-
ments, while the results in Shubik and Levitan (1980) are applicable only to substitutable
products; (ii) We show that the law of demand holds for the demand function under some
mild conditions.

As competition intensifies, more products are offered in markets and their relation-
ships become more complex. Thus good pricing models become core decision tools for
corporations’ revenue management. The next goal of this paper is thus to study a game-
theoretic pricing model in which the above-mentioned demand functions are used. The
best response problem facing each seller is then an NCP constrained optimization prob-
lem. We will show that, in some situations, the NCP constraints in this optimization
problem can be eliminated to obtain a simplified model; the simplification implicitly en-
sures that only $p \in \Omega$ needs to be considered. The computations and analysis are thus
tremendously simplified. As a by-product, this result provides a rigorous justification for
the pricing models introduced in several papers where the demand function is confined
on $\Omega$. However, there are also many other situations in which the aforementioned sim-
plification cannot be realized. Indeed, some examples will be provided in Section 2 to
show that the maximum possible revenue can be achieved at some $p$ outside $\Omega$, when we
solve a single-seller pricing problem. Also in Section 4, we will show, via another exam-
ple, that an equilibrium price obtained from a game involving the linearly constrained
pricing model need not be an equilibrium price of a game involving our LCP constrained
pricing model. Thus in such situations, the complementarity constraints inherited from
the demand functions will remain as a core structure in pricing models.

Summarizing the above, the two main themes of our paper are:

**T1** To achieve a full understanding of a complete, reasonable demand function and
to demonstrate some fundamental properties;

**T2** To introduce a new pricing model, specifically tailored to this demand function,
which forms the basis for and yields an insight into more tractable pricing models.

Finally, we would like to make a remark on different types of demand functions. There
are at least two distinct ways to construct demand functions which are continuous and
nonnegative on $\mathbb{R}^N_+$. The first is directly to find special functions which are smooth and
nonnegative on $\mathbb{R}^N_+$. There are a number of such demand functions, for instance, the
Cobb-Douglas, CES and Logit demand functions, see e.g. Milgrom and Roberts (1990)
and Bernstein and Federgruen (2004). We call these “smooth demand functions”. The
second is to proceed as in the present work: given a smooth function $d$ which is nonnegative
on $\Omega \subset \mathbb{R}^N_+$, we define a function $D$ which agrees with $d$ on $\Omega$ and nonnegative on $\mathbb{R}^N_+$. The
function $D$ is piecewise smooth. Thus, we call the demand functions so constructed
“piecewise smooth demand functions”.

Smooth demand functions look simple and well behaved. However, the requirements
of nonnegativity and smoothness on $\mathbb{R}^N_+$ impose a stringent restriction on the selection of
such functions. Thus, they usually suffer from two major drawbacks. Firstly, they do not
seem to reflect real market behavior: they do not vanish at unacceptably high prices and they tend to unlimited sales at very low prices. Secondly, they are usually not concave, thus constraints like \( q \leq D(p) \) define a nonconvex set of quantities \( q \). This may yield a nonconvex set of optimal solutions and can incur great difficulties in computation and theory. For instance, a pricing game involving such demand functions may not yield a Nash Equilibrium.

Piecewise smooth demand functions may look more complicated. However, the way of constructing piecewise smooth demand functions is much more flexible. One can choose the most suitable function \( d \) which reflects real market behavior and possesses desirable properties, e.g. concavity or linearity, so as to make the theoretical analysis and computation more tractable. Though the piecewise smooth demand function \( D \) is not concave on \( \mathbb{R}^N_+ \) and thus the best response problem in the pricing model is not convex, the NCP structure renders the possibility for the set of optimal solutions of the best response problem to be convex (the proof is not straightforward and will be presented in a separate paper). This property is very important for the existence and computation of equilibrium prices. Furthermore, as will be demonstrated later through Theorems 15 and 16, many such NCP constrained best response problems can be reduced to essentially equivalent but tremendously simplified best response problems. This will dramatically facilitate the computation and theoretical analysis of pricing games.

In the remaining sections of our paper, we will first further discuss the motivation behind our paper in the next section. Then we go on to formulate the model of the demand functions using an NCP, and consider some theoretical properties of the model. Section 3 is devoted to the law of demand. In Section 4, we will introduce a general complementarity constrained best response pricing problem, and discuss the conditions under which the game involving complementarity constrained best response problems can be reduced to a game involving simplified best response problems. Lastly, in the final section, we propose some potential future research directions.

2 Formulation of piecewise smooth demand functions

Suppose that in a market there are \( M \) sellers and seller \( i \) offers \( N_i \) products, for each \( i \in \{1, \ldots, M\} \). In this paper, for simplicity, we assume that all the products offered are distinct. Indeed, in reality, the products offered by different sellers are rarely identical. Thus, there are altogether \( N_1 + \ldots + N_M = N \) distinct products in the market. We denote the price of product \( j \) as \( p_j \) and the price vector of all other products as \( p_{-j} \), for each \( j \in \{1, \ldots, N\} \). Let \( p = (p_1; \ldots; p_N) = (p_j, p_{-j}) \). Let \( D_j \) denote the demand for product \( j \) and \( D = (D_1; \ldots; D_N) \). Here and below, \((x; y; z; \ldots)\) symbolizes a vector in which \( x \) stacks on \( y \), \( y \) stacks on \( z \) and so on. All the vectors in this paper are column vectors. Because the products are substitutable for each other or complementary to each other, demand \( D_j \) depends on the prices of all products, thus it is a function of \( p \).
Let $d : R^N_+ \rightarrow R^N$ be a given function, where $R^N_+ = \{ p \in R^N \mid p \geq 0 \}$. We define $D(p) = d(p)$ for all $p \in \Omega$, where

$$\Omega = \{ p \in R^N_+ \mid d(p) \geq 0 \}.$$ 

If $d$ is linear, then we write 

$$d(p) := b - Ap,$$

where

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix}.$$ 

The diagonal entry $a_{jj}$ of $A$ is the decrement in demand for product $j$ when the price $p_j$ increases by one unit, and the $(j, k)$ entry $a_{jk}$ of $A$ is the amount of demand shifting from product $k$ to product $j$ when the price $p_k$ increases by one unit. Raz and Porteus (2003) discuss some methods used to estimate such parameters facing a single seller.

Implicitly, we assume that the demand for any product in the market only depends on the prices of products sold by the companies in this market, and does not depend on the price of any product sold by the companies outside this market. All other factors that may influence demand are assumed constant and represented by the constant term $b$ or $d(0)$.

It is reasonable to assume that $d(0) \geq 0$ and $d_j(p_j, p_{-j}) < d_j(p'_j, p_{-j})$ if $p_j > p'_j$, given $p_{-j}$ fixed. That is, when all the prices are zero, the demand for each product should be nonnegative and when the price of product $j$ increases, the demand $d_j$ should decrease. In the case of linear $d$, this translates to $b_j \geq 0$ and $a_{jj} > 0$ for all $j = 1, \ldots, N$. But note that products can be substitutable for each other or complementary to each other unless it is specifically indicated, thus e.g., the non-diagonal entries $a_{jk}$ are not restricted.

Now since $D(p) = d(p) \geq 0$ for $p \in \Omega$, $D$ is well defined on $\Omega$. Thus our concern is with the definition of $D$ outside $\Omega$. To explain the strong motivations behind such a study, we will first answer the following question.

2.1 Should demand functions be defined outside $\Omega$?

Because demand functions are building bricks in many OR models, a rigorous and complete model of demand-price relationships is clearly important. Some people may argue that it suffices to consider only prices in $\Omega$ where all components of $d(p)$ are nonnegative, because any price outside $\Omega$ will either create zero demand or be redundant. Although this argument is vague, it may seem acceptable from an applied perspective; and this is probably the reason why the linear demand functions commonly used in the existing literature are confined to the set $\Omega$. Thus, it is worthwhile to provide strong evidence to settle this question.
Example 1 Consider a simple pricing problem involving a single seller offering 2 mutually substitutable products (e.g., business class and economy class air tickets). The demand-price relationships are given by $d_1(p) = 20 - 3p_1 + 2p_2$ and $d_2(p) = 100 + p_1 - 4p_2$. For simplicity, we ignore the costs of production. Let $q_i$ represent the quantity of product $i$ to be sold, where $i = 1, 2$. As it is clear that the quantity sold should be nonnegative and restricted by the amount demanded, we have the constraints $0 \leq q_i \leq d_i(p)$, for $i = 1, 2$. Suppose that the inventory level is 30, i.e., the total amount sold is restricted to be not more than 30. This translates to the constraint $q_1 + q_2 \leq 30$. Due to differences in quality or attributes of the 2 products, the price of product 1 is restricted to be not less than that of product 2. This gives rise to the last constraint $p_1 \geq p_2 \geq 0$. The problem is then to decide on the prices to set (and the amount to sell) to maximize the seller’s revenue $q_1p_1 + q_2p_2$, given the demand, inventory and pricing constraints above.

Suppose we do not allow the consideration of prices $p \notin \Omega$. Then the constraints $p_1 \geq p_2$ and $d_1(p) \geq 0$ imply that

$$p_1 \leq 3p_1 - 2p_2 \leq 20.$$ 

Thus, the maximum revenue that can be obtained is bounded from above by

$$q_1p_1 + q_2p_2 \leq q_1p_1 + q_2p_1 \leq 30p_1 \leq 30 \times 20 = 600.$$ 

However, if we consider the possibility of setting prices outside $\Omega$, we can show that a higher revenue can be obtained. Now we denote by $D(p)$, the demand function on $R^+_N$ such that $D(p) = d(p)$ for $p \in \Omega$, and $D(p)$ satisfies the following reasonable conditions outside $\Omega$: (i) $D_i(p) = 0$ if $d_i(p) < 0$, and (ii) $D_i(p_i, p_j)$ is nondecreasing in other product’s price $p_j$ (here we assume that products are substitutable).

If we set $p_2 = 23$, then at any $p_1 \geq 22, d_1(p) \leq 0$. For example, if we set the price to be $\bar{p} = (24, 23)$, then we should have by (i), $D_1(\bar{p}) = 0$ and by (ii), noting that $(22, 23) \in \Omega$, $D_2(\bar{p}) \geq D_2(22, 23) = d_2(22, 23) = 100 + 22 - 4 \times 23 = 30$. Choosing to sell 30 units of product 2 and none of product 1, i.e., $q_1 = 0, q_2 = 30$, it is easy to see that all constraints are met and the total revenue obtained is

$$q_1p_1 + q_2p_2 = 0 + 30 \times 23 = 690 > 600.$$ 

This example shows clearly that at times, we do need to consider prices outside $\Omega$ to achieve higher profits.

Let us consider another simple example.

Example 2 This example is similar to Example 1, with the exception that the pricing constraints $p_1 \geq p_2 \geq 0$ are replaced by $p_1 \geq 16, p_2 \geq 32$.

Now if we can only set prices in $\Omega$, then $d_1(p) = 20 - 3p_1 + 2p_2 \geq 0$ and $d_2(p) = 100 + p_1 - 4p_2 \geq 0$ imply that we must have $(p_1, p_2) \leq (28, 32)$. Since $p_2 \geq 32$ is
necessary, it is easy to see that the only possible prices to set are \( p_1 = 28 \) and \( p_2 = 32 \). Correspondingly, \( d_1 = d_2 = 0 \). That is, no revenue can be obtained in this case.

However, if we allow prices outside \( \Omega \) to be considered, we can easily obtain some revenue by simply setting \((p_1, p_2) = (16, 32)\). Now at \( p_1 = 16 \), by a similar reasoning to the above, for any \( p_2 \geq 29 \), \( D_2(16, p_2) = 0 \) and \( D_1(16, p_2) \geq D_1(16, 29) = d_1(16, 29) = 30 \). Choosing to sell 30 units of product 1 and none of product 2 at the price \((p_1, p_2) = (16, 32)\), all constraints are satisfied and we obtain a revenue of \( p_1q_1 = 16 \times 30 = 480 \).

This example was specially chosen for simplicity of presentation. However, it is not difficult, though it is more complicated, to check that a similar phenomenon exists for many other lower bound constraints on prices. For example, the constraints can be \((p_1, p_2) \geq (25, 25)\) or \((p_1, p_2) \geq (15, 30)\).

In Section 4, we will present another example which involves a two-seller pricing game. That example has the following important consequence: an equilibrium price obtained from a game where sellers are only allowed to set prices in \( \Omega \) need not be an equilibrium price if prices outside \( \Omega \) are allowed.

Having understood the necessity of a complete definition of the demand function on \( \mathbb{R}^N_+ \), we are now faced with the problem of actually defining the demand function.

### 2.2 How to define demand functions outside \( \Omega \).

For the single-product case, the demand function of the form

\[
D(p) = \begin{cases} 
  d(p) & \text{if } p \in \Omega \\
  0 & \text{if } p \notin \mathbb{R}^N_+ \setminus \Omega
\end{cases}
\]

is most natural and commonly used. We attempt to extend it to the multiple-product case.

The function \( d_i \) can be negative or positive at a \( p \notin \Omega \). It is natural to define \( D_i(p) = 0 \) if \( d_i(p) < 0 \). The difficulty lies in the setting of value of \( D_i(p) \) if \( d_i(p) > 0 \) at \( p \notin \Omega \). The following are possible and apparently natural ways to model demand outside \( \Omega \).

A possible model can be

\[
D(p) = \begin{cases} 
  d(p) & \text{if } p \in \Omega \\
  0 & \text{if } p \notin \mathbb{R}^N_+ \setminus \Omega
\end{cases}
\]

However, for \( \bar{p} \in \partial \Omega \) (the boundary of \( \Omega \)), we have \( d_i(\bar{p}) = 0 \) for some component \( i \), but we may also have \( d_j(\bar{p}) > 0 \) for another component \( j \). Since \( D_j(p) = 0 \) for all \( p \notin \Omega \), \( D \) so defined cannot be continuous, which is clearly undesirable.
To avoid the discontinuity, we can consider another model. For each \( j \in \{1, \ldots, N\} \),

\[
D_j(p) = \begin{cases} 
d_j(p) & \text{if } d_j(p) \geq 0 \\
0 & \text{if } d_j(p) < 0.
\end{cases}
\]

A consequence of this model is: if \( d_j(\bar{p}) = 0 \), then for any \( p \) with \( p_j > \bar{p}_j \) (with other prices unchanged), \( D_j(p) = 0 \) since \( d_j(p) < 0 \), but for each product \( i (i \neq j) \) that is substitutable for product \( j \), \( D_i(p) (= d_i(p) > 0) \) increases and tends to infinity as \( p_j \to \infty \). That is, the demand for product \( i \) increases to infinity when its own price \( p_i \) is unchanged and some other price \( p_j \) tends to infinity. This is very unlikely to be valid in the market.

Now, what should a rational demand function satisfy?

Consider a market of two substitutable products with demands \( D_1 \) and \( D_2 \) and prices \( p_1 \) and \( p_2 \). Fix \( p_1 = \bar{p}_1 \) and increase \( p_2 \). At the beginning, assuming that both \( D_1 \) and \( D_2 \) are positive, i.e. \( p = (p_1, p_2) \in int(\Omega) \) (the interior of \( \Omega \)), the demand \( D_2 \) decreases as \( p_2 \) increases because some customers stop purchasing product 2. These customers may decide to buy product 1 instead, thus \( D_1 \) may increase. After \( p_2 \) is increased to a certain value \( \bar{p}_2 \), the demand \( D_2 \) will drop to zero (assuming \( \Omega \) is bounded). At this price, no customer purchases product 2 any more. Any further increase of \( p_2 \) should not affect the market because \( D_2 \) will remain at zero and \( D_1 \) will not increase any more, as no customer demand can shift from product 2 to product 1. Based on this observation, a proper demand function \( D \) should satisfy \( D(\bar{p}_1, p_2) = D(\bar{p}_1, \bar{p}_2) \) for any \( p_2 > \bar{p}_2 \). This property is formally stated below.

**Definition 1** A demand function \( D \) that maps \( R^N_+ \) into \( R^N_+ \) is said to be regular if \( D_i(\bar{p}) = 0 \) and \( \Delta \geq 0 \), then \( D(\bar{p} + \Delta e_i) = D(\bar{p}) \) where \( e_i \) is the \( i \)-th unit vector.

A regular demand function is defined as a demand function in Shubik and Levitan (1980, Appendix B). More justifications for the regularity can be found there.

In the above illustration, we see that \( \bar{p} = (\bar{p}_1, \bar{p}_2) \) is on \( \Omega \) (more precisely, on the boundary of \( \Omega \)). For any \( p = (\bar{p}_1, p_2) \) outside \( \Omega \) with \( p_2 > \bar{p}_2 \), we see that \( D(p) = D(\bar{p}) = d(\bar{p}) \). This observation suggests that we can first define a map \( B(p) = \bar{p} \) and then define \( D(p) = d(B(p)) \). Such a map \( B \) can be defined via a complementarity problem as follows.

**Definition 2** For any \( p \in R^N_+ \), \( B(p) \) is defined as the solution of the NCP(\( p \)): find \( x (= B(p)) \) such that

\[
0 \leq d(x) \perp p - x \geq 0, \tag{3}
\]

where \( \perp \) stands for perpendicular and \( d(x) \perp p - x \iff d(x)^T(p - x) = 0. \)

Definition 2 is essentially equivalent to Problem 1 in Shubik and Levitan (1980, Appendix B). We formulate it explicitly as an NCP (Nonlinear Complementarity Problem),
so that the rich results available in the well-established area of NCPs can be utilized and applications of this demand function can be more easily investigated.

Throughout this paper we make the following assumption.

**Assumption 1.** For all \( p \in R_+^N \), the NCP\((p)\) has a unique solution \( B(p) \) and \( B(p) \in \Omega \).

There has been extensive research on the existence and uniqueness of solutions to complementarity problems, cf. Cottle, Pang and Stone (1992) or Facchinei and Pang (2003). If \( d \) is linear, a necessary and sufficient condition for the existence of a unique solution to the LCP (Linear Complementarity Problem) is that \( A \) is a P-matrix (i.e., all the principal minors of \( A \) are positive). Thus in this paper, we assume that \( A \) is a P-matrix. For nonlinear \( d \), if \( d \) is a P-function\(^1\), then the NCP\((p)\) has at most one solution, and if \( d \) is a uniformly P-function\(^2\), then the NCP\((p)\) has a unique solution for any \( p \in R_+^N \). These conditions also ensure that the map \( B \) is continuous and piecewise smooth provided \( d \) is smooth, cf. Facchinei and Pang (2003). However, the conditions that can ensure \( B(p) \in \Omega \) is less obvious. We will illustrate this property for the linear function \( d \) in Lemma 6.

Under Assumption 1, we can define a demand function.

**Definition 3** The demand function \( D : R_+^N \to R_+^N \) is defined by

\[
D(p) = d(B(p)), \quad \forall p \in R_+^N,
\]

where the map \( B \) is as stated in Definition 2.

The nonnegativity of \( D \) follows directly from the facts that \( B(p) \in \Omega \) (as assumed) and \( d \) is nonnegative on \( \Omega \). In addition, since \( d \) and \( B \) are continuous, the continuity of \( D \) is obvious.

Now, an economic motivation of this demand function is as follows.

A typical approach to deriving a demand function is to maximize a utility function subject to the consumers’ income and a nonnegative purchase quantity constraint, see for example, Part one of the classic book on microeconomic theory by Mas-Colell, Whinston and Green (1995). By maximizing an appropriate quadratic utility function, this approach

\(^1\) \( F \) is a P-function on \( R_+^N \) if for all pairs of distinct vectors \( x \) and \( y \) in \( R_+^N \),

\[
\max_{1 \leq i \leq N} (x_i - y_i)(F_i(x) - F_i(y)) > 0.
\]

\(^2\) \( F \) is a uniformly P-function on \( R_+^N \) if there exists a constant \( \mu > 0 \) such that for all pairs of vectors \( x, y \) in \( R_+^N \),

\[
\max_{1 \leq i \leq N} (x_i - y_i)(F_i(x) - F_i(y)) \geq \mu \|x - y\|_2^2.
\]
can derive the above demand function through the optimality conditions if $d$ is linear and the matrix $A$ is symmetric. See for example, Dixit (1979), Shubik and Levitan (1980, Chapter 7), and Eliashberg and Steinberg (1991), for a similar derivation (where the linear demand function with symmetric $A$ was considered). Although this approach is not applicable to our general model, it exhibits an economic insight into the demand model.

In addition, this demand function is a unique extension of the function $d$, as shown below.

**Theorem 4** Under Assumption 1, the function $D$ as defined in Definition 3 is the unique regular demand function which agrees with $d$ on $\Omega$.

This theorem is a slight modification of Theorem 1 in Shubik and Levitan (1980, Appendix B). Thus, the proof of Theorem 1 in the book is applicable here.

Now let us discuss the conditions which can guarantee that the assumption $B(p) \in \Omega$ for any $p \in R_+^N$ holds true. For the linear function $d$, we can present a necessary and sufficient condition. For nonlinear $d$, the conditions required are more intricate. Thus, we will illustrate this only for linear $d$. First we need the following result which shows when $\Omega$ is bounded.

**Lemma 5** If $d$ is linear and $A$ is a P-matrix, then $\Omega$ is bounded.

**Proof.** If $\Omega$ is not bounded, then there exists a point $\bar{p} \in \Omega$ and a direction $u \geq 0$, such that

$$d(\bar{p} + \lambda u) = b - A(\bar{p} + \lambda u) \geq 0, \quad \forall \lambda \geq 0.$$ 

that is,

$$d(\bar{p}) - \lambda Au \geq 0, \quad \forall \lambda \geq 0.$$ 

Let

$$J = \{j \mid u_j > 0\}.$$ 

Note that $J \neq \emptyset$. Consider the rows $d_j$ with $j \in J$,

$$d_J(\bar{p}) - \lambda A_{J,J}u_J \geq 0, \quad \forall \lambda \geq 0.$$ 

This implies

$$A_{J,J}u_J \leq 0.$$ 

Since $u_J > 0$, we have

$$u_j(A_{J,J}u_J)_j \leq 0, \quad \forall j \in J.$$ 

Because $A_{J,J}$ is a P-matrix, by Theorem 3.3.4 (b) in Cottle, Pang and Stone (1992), this implies $u_J = 0$, which is a contradiction since by definition $u_J > 0$.

For any index set $K \subseteq \{1, 2, \ldots, N\}$, we always denote by $\bar{K}$ the complement of $K$, i.e. $\bar{K} = \{1, 2, \ldots, N\} \setminus K$. 

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Lemma 6 Let $d$ be linear and $A$ be a $P$-matrix. Then, $B(p) \in \Omega$ for all $p \in R^N_+$ if and only if
\[ A_{J}^{-1}b_J \geq 0, \quad \forall J \subseteq \{1, 2, \ldots, N\}. \] (4)

More precisely, under the above condition, $B(p) = p$ if $p \in \Omega$ and $B(p) \in \partial \Omega$ if $p \in R^N_+ \setminus \Omega$.

Proof. “$\Leftarrow$” Suppose condition (4) holds. We want to show that $x = B(p) \in \Omega$. Since $LCP(p)$ implies $b - Ax \geq 0$, we need only to show $x \geq 0$. Let $K = \{k \mid d_k(x) = 0\}$ and $\bar{K} = \{k \mid d_k(x) > 0\}$. Then by the complementarity conditions of $LCP(p)$, we have $p_K - x_K = 0$. Since $p \geq 0$, we have $x_K \geq 0$.

To show $x_K \geq 0$, we consider the polyhedron
\[ H = \{z \mid d_K(z) = 0, \, d_\bar{K}(z) \geq 0, \, z_K \geq 0\}. \]

Obviously, $x \in H$. Any vertex $y$ of $H$ is associated with an index set $J$ such that $J \supseteq K$ and
\[ d_J(y) = 0, \quad y_J = 0. \]

This implies $y_J = A_{J}^{-1}b_J \geq 0$ and shows that every vertex $y$ of $H$ is nonnegative. Since $x = B(p)$ can be represented as a convex combination of the vertices of $H$, $B(p) \geq 0$.

“$\Rightarrow$” For any $K \subseteq \{1, 2, \ldots, N\}$, we choose a $p \in R^N_+$ with $p_K = 0$ and $p_\bar{K} \geq 0$ sufficiently large such that $p_K > x_K$ for all $x \in \Omega$. Such a $p_K$ exists because $\Omega$ is bounded by Lemma 5.

Let $\bar{p} = B(p)$. By assumption, $B(p) \in \Omega$. Thus, $\bar{p} \geq 0$.

Since $p \geq \bar{p} \geq 0$ and $p_K = 0$, we have $\bar{p}_K = 0$.

It follows from $p_K > \bar{p}_K$ and $(p_K - \bar{p}_K)^T d_K(\bar{p}) = 0$ that $d_K(\bar{p}) = 0$. This yields $\bar{p}_K = A_{K}^{-1}b_K$. Thus $\bar{p}_K \geq 0$ implies $A_{K}^{-1}b_K \geq 0$. \qed

Remark: If $d$ is linear and we are considering a system of $N$ mutually substitutable products, then all the off-diagonal entries of $A$ will be non-positive. If in addition, $A$ is a $P$ matrix, then $A$ is an $M$-matrix. Thus the inverse of $A$ and all its principal submatrices are nonnegative (see Berman and Plemmons (1994)). It follows that with $b \geq 0$, $A_J^{-1}b_J \geq 0$ will always be satisfied for all $J \subseteq \{1, 2, \ldots, N\}$. This special case was considered in Shubik and Levitan (1980).

The map $B$ has a simple geometric structure. To show this, we introduce some notations:
\[ J(\bar{p}) = \{j \mid d_j(\bar{p}) = 0\} \subseteq \{1, 2, \ldots, N\} \]
and the cone
\[ C(\bar{p}) = \{p \mid p_j \geq \bar{p}_j \forall j \in J(\bar{p}); \, p_i = \bar{p}_i \forall i \notin J(\bar{p})\}. \] (5)

Notice that $C(\bar{p}) = \{\bar{p}\}$ for $\bar{p}$ in the interior of $\Omega$ because $J(\bar{p}) = \emptyset$. 

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Lemma 7 Under Assumption 1,

(i) \( p \in C(B(p)) \) for any \( p \in \mathbb{R}^N_+ \);

(ii) If \( \bar{p} \in \Omega \) and \( p \in C(\bar{p}) \), then \( \bar{p} = B(p) \).

Proof. (i) Denote \( J(p) = \{ j \mid d_j(p) = 0 \} \). Let \( J = J(B(p)) \) and \( \bar{J} = \{1, 2, \ldots, N\} \setminus J \).

For any \( p \in \mathbb{R}^N_+ \), under Assumption 1, \( B(p) \in \Omega \) (see Lemma 6 for linear \( d \)). By the definition of \( B \), \( p \geq B(p) \) and \( p_j = B(p)_j \). This shows that \( p \in C(B(p)) \).

(ii) For any \( \bar{p} \in \Omega \) and \( p \in C(\bar{p}) \), we have \( p \geq \bar{p} \) and \( d(\bar{p}) \geq 0 \). Let \( J = J(\bar{p}) \) and \( \bar{J} = \{1, 2, \ldots, N\} \setminus J \). Then by definition of \( C(\bar{p}) \), \( p_j = \bar{p}_j \) and \( d_j(\bar{p}) = 0 \). This yields \( (p - \bar{p})^T d(\bar{p}) = 0 \). Thus, \( \bar{p} \) is a solution of \( \text{NCP}(p) \). Under Assumption 1, the solution of \( \text{NCP}(p) \) is unique, which is \( B(p) \). Thus, \( \bar{p} = B(p) \).

Lemma 7 shows that for any \( p \in \mathbb{R}^N_+ \), there exists a unique \( \bar{p} \in \Omega \) such that \( p \in C(\bar{p}) \). In fact, \( \bar{p} = B(p) \). This means that the map \( B(p) \) can be equivalently defined by the cone \( C(p) \).

We now have both algebraic characterization (by LCP) and geometric characterization (by the cones \( C(p) \)) of the map \( B \).

It may be interesting to investigate other possible choices of the map. Hence, we briefly digress from our focus on Theme (T1). One may have thought of using the orthogonal map instead of \( B \). However, we will show that the orthogonal map is not suitable.

Example 3 Suppose \( N(p) \) is the orthogonal projection of \( p \) on \( \Omega \). Let us consider a single seller offering 2 products that are mutually substitutable (\( M = 1 \) and \( N = N_1 = 2 \)). Let \( \bar{p} = (\bar{p}_1, \bar{p}_2) \in \partial \Omega \) with \( d_1(\bar{p}) = 0 \) and \( d_2(\bar{p}) > 0 \) (where \( d_1, d_2 \) are linear). Consider \( p' = (p_1, p_2) \) with \( p_1 > \bar{p}_1 \). The orthogonal projection, \( N(p') \), of \( p' \) on \( d_1(p) = 0 \) satisfies \( N(p') > \bar{p} \), and hence \( d_2(N(p')) < d_2(\bar{p}) \). See Figure 1. If we define \( D(p) = d(N(p)) \), then \( D_2(p_1, \bar{p}_2) < D_2(\bar{p}_1, \bar{p}_2) \) for \( p_1 > \bar{p}_1 \). This means that with \( p_2 \) kept constant, an increase in \( p_1 \) leads to a decrease in demand for product 2. However, one would expect either an increase or no change in the demand for product 2, resulting from either a shift in demand from product 1 to product 2, or a decision to give up on buying. That is, it is not reasonable to have \( D_j \) decreasing in \( p_k \), for \( k \neq j \), if we are considering a market of mutually substitutable products. Therefore, the orthogonal map is not suitable for defining the demand function.
To complete the discussion on Theme (T1) of our paper, we will now study some desirable and useful properties of the demand function.

3 The law of demand

The demand function defined in the preceding section possesses some properties which are desirable from the perspective of Economics and Management. In this paper, we will only investigate a fundamental property: the law of demand.

The law of demand is a fundamental law in economics. It states that the price and demand of a good pull in opposite directions. Mathematically, this is referred to as the monotonicity of demand functions. It is a desirable property in many applications of demand functions.

The law of demand is ingrained in our everyday life. Shoppers buy more strawberries when they are in season and the price is lower. Similarly, when people learn that frost will strike orange groves in Florida, they know that the price of orange juice will rise. Higher prices are set in order to reduce the amount demanded. This is the law of demand.

We will first investigate the change of demand of a product as its own price varies, then followed by the change of demand of all the products as all prices vary. Although the former is a special case of the latter, the former may be observed under weaker conditions. Note that most of the results in this section are proven for the linear $d$ case for simplicity.

**Theorem 8** Let $d$ be linear. If $A$ is a $P$-matrix and the condition (4) is satisfied, then the demand function $D : R^N_+ \to R^N_+$ defined in Definition 3 satisfies: $D_j$ is non-increasing in $p_j$. 

Figure 1. Illustration of orthogonal projection $N$ and mapping $B$. 
Proof. Suppose that the price vector of all the products excluding product \( j \), \( p_{-j} \), is fixed. So we can consider \( D_j \) and \( B \) as functions of \( p_j \), and write \( D_j(p_j) \) and \( B(p_j) \) instead. We first consider the interval

\[ \Omega_j := \{ p_j \mid D_j(p_j) > 0 \}. \]

For every \( p_j \in \Omega_j \), we denote \( K := \{ k \mid d_k(B(p_j)) = 0 \} \) and \( \bar{K} = \{1, ..., N\} \setminus (K \cup \{ j \}) \). Then since \( b_K - A_{Kj}B_j(p_j) - A_{K\bar{K}}B_{\bar{K}}(p_j) - A_{K\bar{K}}B_{\bar{K}}(p_j) = 0 \) and \( B_j(p_j) = p_j \) and \( B_{\bar{K}}(p_j) = p_{\bar{K}} \) by complementarity, we have

\[ B_K(p_j) = A_{K\bar{K}}^{-1}(b_K - A_{Kj}p_j - A_{K\bar{K}}p_{\bar{K}}). \]

Thus,

\[
D_j(p_j) = b_j - A_{jj}p_j - A_{jK}p_K - A_{jK}A_{K\bar{K}}^{-1}(b_K - A_{Kj}p_j - A_{K\bar{K}}p_{\bar{K}}) = (b_j - A_{jK}p_K - A_{jK}A_{K\bar{K}}^{-1}b_K + A_{jK}A_{K\bar{K}}^{-1}A_{K\bar{K}}p_{\bar{K}}) - (A_{jj} - A_{jK}A_{K\bar{K}}^{-1}A_{K\bar{K}}) p_j.
\]

Since \( D_j \) is continuous, \( D_j \) is non-increasing on \( \Omega_j \) if and only if the term \( A_{jj} - A_{jK}A_{K\bar{K}}^{-1}A_{K\bar{K}} \) is nonnegative at every \( p_j \in \Omega_j \). Let \( \alpha = \{ j, K \} \). Then \( A_{\alpha\alpha} = \begin{pmatrix} A_{jj} & A_{jK} \\ A_{Kj} & A_{K\bar{K}} \end{pmatrix} \). Now we have

\[
\det A_{\alpha\alpha} = \det \begin{pmatrix} A_{jj} - A_{jK}A_{K\bar{K}}^{-1}A_{Kj} & 0 \\ A_{Kj} & A_{K\bar{K}} \end{pmatrix} = (A_{jj} - A_{jK}A_{K\bar{K}}^{-1}A_{Kj}) \det A_{K\bar{K}}.
\]

Since \( A \) is a P-matrix, both \( \det A_{\alpha\alpha} \) and \( \det A_{K\bar{K}} \) are positive. Thus,

\[ A_{jj} - A_{jK}A_{K\bar{K}}^{-1}A_{Kj} > 0. \]

After \( D_j(p_j) \) decreases to 0, as \( p_j \) increases further, \( D_j(p_j) \) remains at 0. Thus \( D_j \) is non-increasing in \( p_j \).

The above result is a very basic property that \( D \) should satisfy. However, we must also consider the case in which the prices of more than one product vary.

Now, the law of demand is well observed in the case of a single product. For a market of multiple differentiated products, a similar phenomenon has also been observed. A non-rigorous statement is that total demand for all the products should not increase for any increase in prices.

The difficulty of extending the law of demand to multiple-product markets lies in that the space of \( \mathbb{R}^N \) is not totally ordered, i.e., not every pair of points can be ordered. For instance, we cannot tell which of \((0, 1)\) and \((1, 0)\) is larger.

We can express the law of demand for a single product (the demand decreases (not necessarily strictly) if the price increases) in two different ways:

\[
\text{(i) } D(p') \leq D(p), \quad \forall p' \geq p > 0;
\]
To extend (i) to the multiple-product market, we consider the total demand \( \eta^T D(p) \) (with weight \( \eta \in R^N \) ), i.e.

\[
\eta^T D(p') \leq \eta^T D(p), \quad \forall p' \geq p \geq 0.
\]

The extension of (ii) to the multiple-product market is known as the Law of Demand in Mas-Colell, Whinston and Green (1995) \(^3\). More explicitly, it is given by

\[
(p' - p)^T (D(p') - D(p)) \leq 0, \quad \forall p', p \in R^N_+.
\]

We will show that the demand model \( D \) satisfies these properties. However, we first need the technical lemma below.

**Lemma 9** For any index set \( J \subseteq \{1, 2, \ldots, N\} \), if \( d \) is linear, the set

\[
S_J = \{ p \in R^N_+ \mid d_J(B(p)) = 0, \quad p_J = B(p)_J \}
\]

is convex, and the restriction of the map \( B \) on \( S_J \) satisfies

\[
B(\lambda p^1 + (1 - \lambda)p^2) = \lambda B(p^1) + (1 - \lambda)B(p^2), \quad \forall p^1, p^2 \in S_J, \lambda \in [0, 1]. \quad (6)
\]

**Proof.** Let \( p^1, p^2 \in S_J \). Denote

\[
p(\lambda) = \lambda p^1 + (1 - \lambda)p^2,
\]

and let

\[
x(\lambda) = \lambda B(p^1) + (1 - \lambda)B(p^2).
\]

It follows from \( d_i(B(p^1)) \geq 0, i = 1, 2 \), that

\[
d(x(\lambda)) \geq 0, \quad \forall \lambda \in [0, 1]. \quad (7)
\]

It follows from \( p^i \geq B(p^i), i = 1, 2 \), that

\[
p(\lambda) \geq x(\lambda), \quad \forall \lambda \in [0, 1]. \quad (8)
\]

Furthermore, since \( p^i \in S_J, i = 1, 2 \), one can verify that

\[
d_J(x(\lambda)) = \lambda d_J(B(p^1)) + (1 - \lambda)d_J(B(p^1)) = 0, \quad \forall \lambda \in [0, 1]. \quad (9)
\]

\(^3\)More precisely, the book Mas-Colell, Whinston and Green (1995) distinguishes *Compensated* from *Uncompensated* Law of Demand. In this paper, the wealth of buyers is assumed fixed, thus the Law of Demand stated here is the Uncompensated Law of Demand in Mas-Colell, Whinston and Green (1995).
and
\[ p(\lambda)_j = x(\lambda)_j, \quad \forall \lambda \in [0, 1]. \quad (10) \]

It follows from (9) and (10) that
\[ d(x(\lambda)) \perp (p(\lambda) - x(\lambda)). \quad (11) \]

Now, (7), (8) and (11) show that \( x(\lambda) \) is the solution of the LCP(\( p(\lambda) \)), thus
\[ x(\lambda) = B(p(\lambda)). \quad (12) \]

Substituting (12) into (9) and (10), we see that \( p(\lambda) \in S_J \). This shows the convexity of \( S_J \).

The equation (6) follows from (12).

**Theorem 10** Let \( d \) be linear and \( \eta \in \mathbb{R}^N \). If \( A \) is a P-matrix, the condition (4) is satisfied, and for any \( J \subseteq \{1, 2, \ldots, N\} \) (\( J \) can be \( \emptyset \)) and its complement \( \bar{J} = \{1, 2, \ldots, N\} \setminus J \),
\[ \eta_J^T (A_{jj} - A_{jj}A_{jj}^{-1}A_{jj}) \geq 0, \quad (13) \]
then the demand function \( D : \mathbb{R}^+_N \rightarrow \mathbb{R}^+_N \) defined in Definition 3 exhibits the following property:
\[ \eta^T D(p') \leq \eta^T D(p), \quad \forall p' \geq p \geq 0, \]
that is, the total demand (with weight \( \eta \)) does not increase if no price decreases.

**Proof.** For any \( p, p' \) in \( \mathbb{R}^+_N \) such that \( p' \geq p \), let \( x = B(p) \) and \( x' = B(p') \). By Assumption 1, \( x, x' \in \Omega \).

Firstly, consider \( p \) and \( p' \) which satisfy
\[ d_J(x) = d_J(x') = 0, \quad x_j = p_j, \quad x'_j = p'_j, \quad (14) \]
for some \( J \subseteq \{1, 2, \ldots, N\} \).

If \( J = \emptyset \), then \( x = p \) and \( x' = p' \). Thus,
\[ \eta^T (D(p') - D(p)) = \eta^T (d(p') - d(p)) \]
\[ = -\eta^T A(p' - p) \]
\[ \leq 0, \]
where the last inequality holds because \( \eta^T A \geq 0 \) by condition (13), and \( p' - p \geq 0 \).
Now we assume $J \neq \emptyset$. From (14), it follows that

$$x'_J - x_J = -A_j^{-1}A_{jj}(p'_J - p_J)$$

Then

$$\eta^T(D(p') - D(p)) = \eta^T(d(x') - d(x)) = \eta_j^T(d_j(x') - d_j(x)) = -\eta_j^T(A_{jj}(x'_J - x_J) + A_{jj}(p'_J - p_J)) = -\eta_j^T(A_{jj} - A_{jj}A_{jj}^{-1})(p'_J - p_J).$$

Since $(p'_J - p_J)$ is an arbitrary nonnegative vector, $\eta^T(D(p') - D(p)) \leq 0$ if and only if the condition (13) holds.

Now for any $p \leq p'$ in $R^N_+$, denote $p(t) = p + t(p' - p)$. Then we shall show $\eta^T D(p(0)) \geq \eta^T D(p(1))$. By the complementarity conditions, at any $\hat{p} \in R^N_+$, there is a $\hat{J}$ ($\hat{J}$ can be $\emptyset$) and $J = \{1, 2, \ldots, N\} \setminus \hat{J}$ such that

$$D_J(\hat{p}) = 0, \quad \hat{p}_J = \hat{x}_J$$

hold.

Thus, we can partition the interval $[0, 1]$ into several sub-intervals $[t_i, t_{i+1}] \subseteq [0, 1]$, on which the following condition holds for an index set $J_i$ ($J_i$ can be $\emptyset$) and $\bar{J}_i = \{1, 2, \ldots, N\} \setminus J_i$:

$$D_{J_i}(p(t)) = 0, \quad p(t)_{J_i} = x(t)_{J_i}, \quad \forall t \in [t_i, t_{i+1}].$$

Now we shall show that the number of sub-intervals is finite.

We say that two adjacent sub-intervals $[t_{i-1}, t_i]$ and $[t_i, t_{i+1}]$ can be merged if there exists an index set $K$ such that

$$d_K(x(t)) = 0, \quad p(t)_K = x(t)_K, \quad \forall t \in [t_{i-1}, t_{i+1}].$$

We assume the above partition is the least partition, i.e. no adjacent sub-intervals can be merged. Then, we can show all index sets $J_i$ are distinct.

By contradiction, suppose there are $k < l$ with $J_k = J_l = J$. For any $t \in [t_k, t_{i+1}]$, let $\lambda \in [0, 1]$ be such that $t = \lambda t_k + (1 - \lambda)t_{i+1}$. Then

$$p(t) = p + t(p' - p) = \lambda p(t_k) + (1 - \lambda)p(t_{i+1}).$$

Since $p(t_k), p(t_{i+1}) \in S_J$ and $S_J$ is convex by Lemma 9, we have $p(t) \in S_J$ for any $t \in [t_k, t_{i+1}]$. Since $k \leq l - 1 < l$, by definition of $S_J$, we have

$$d_J(x(t)) = 0, \quad p(t)_J = x(t)_J, \quad \forall t \in [t_{i-1}, t_{i+1}].$$
This shows that the sub-intervals \([t_{l-1}, t_l]\) and \([t_l, t_{l+1}]\) can be merged, and the partition is not the least partition, which is a contradiction. Therefore, all index sets \(J_i\) must be distinct, and thus the number of sub-intervals must be finite.

For any two points on the line segment connecting \(p(t_i)\) and \(p(t_{i+1})\), the condition (14) holds for \(J_i\). Thus, as shown above, we have \(\eta^T D(p(t_i)) \geq \eta^T D(p(t_{i+1}))\). This leads to \(\eta^T D(p) \geq \eta^T D(p')\).

If the market we consider consists of mutually substitutable products, then the following result follows.

**Corollary 11** If \(d\) is linear and \(A\) is a weighted column dominant (i.e., \(\eta^T A \geq 0\), where \(\eta \in \mathbb{R}^N\)) P-matrix with \(a_{ij} \leq 0\) \(\forall i, j \in \{1, \ldots, N\}, j \neq i\) and \(b \geq 0\), then \(\eta^T D(p') - \eta^T D(p) \leq 0\), \(\forall p' \geq p \geq 0\).

**Proof.** Firstly, if \(A\) is a P-matrix with non-positive off-diagonal entries and \(b \geq 0\), Assumption 1 is satisfied.

Secondly, we observe that with \(Q = \begin{pmatrix} I & -A_{JJ}^{-1}A_{JJ} \end{pmatrix} \), \(AQ = \begin{pmatrix} A_{JJ} & 0 \\ A_{JJ}^{-1}A_{JJ} & I \end{pmatrix}\). Since \(A\) is weighted column dominant, \(\eta^T A \geq 0\). By our hypothesis on \(A\), \(A\) is an M-matrix, i.e., \(A_{JJ}^{-1} \geq 0\) (all entries are nonnegative), and thus \(Q \geq 0\). Then \(\eta^T AQ \geq 0\) follows and \(\eta^T (A_{JJ}^{-1}A_{JJ} - A_{JJ}A_{JJ}^{-1}A_{JJ}) \geq 0\).

All conditions in Theorem 10 are satisfied. The proof is completed by using Theorem 10.

Theorem 10 and Corollary 11 show that \(\eta^T D(p)\) decreases as \(p\) increases. This seems to be a natural extension of the law of demand from a one-product market to a multiple-product market. However, since \(p \in \mathbb{R}^N_+\) is not totally ordered, the condition that \(p\) increases covers only a very limited situation. Moreover, expressing the monotonicity of a vector \(D(p)\) by the monotonicity of \(\eta^T D(p)\) also sacrifices generality.

The second inequality

\[
(p' - p)^T (D(p') - D(p)) \leq 0, \quad \forall p, p' \in \mathbb{R}^N_+
\]

is a more appropriate expression of monotonicity for a multiple-product system. Moreover, this monotonicity property of \(D\) is desirable in theoretical analysis and has been used as a condition in many research papers, see for example, Perakis and Sood (2004).

**Theorem 12** Under Assumption 1, if \(d\) is monotonically decreasing on \(\mathbb{R}^N_+\), then the demand function \(D : \mathbb{R}^N_+ \rightarrow \mathbb{R}^N_+\) defined in Definition 3 is monotonically decreasing. That is,

\[
(p' - p)^T (D(p') - D(p)) \leq 0, \quad \forall p, p' \in \mathbb{R}^N_+.
\]
Proof. For any $p, p' \in R_+^N$, denote $x = B(p)$ and $x' = B(p')$. We will first prove that

$$((p' - x') - (p - x)) \cdot (D(p') - D(p)) \leq 0, \quad \forall p, p' \in R_+^N. \quad (15)$$

Now

$$((p' - x') - (p - x)) \cdot (D(p') - D(p)) = (p' - x')^T D(p') - (p' - x')^T D(p)$$

$$- (p - x)^T D(p') + (p - x)^T D(p)$$

$$= (p' - x')^T d(x') - (p' - x')^T d(x)$$

$$- (p - x)^T d(x') + (p - x)^T d(x)$$

$$= -(p' - x')^T d(x) - (p - x)^T d(x')$$

$$\leq 0,$$

where the third equality follows from the complementarity constraints $(p' - x')^T d(x') = (p - x)^T d(x) = 0$, and the inequality is due to the nonnegativity constraints on $p' - x', p - x, d(x')$ and $d(x)$. Thus (15) is true.

This implies that $\forall p, p'$,

$$(p' - p)^T (D(p') - D(p)) \leq (x' - x) \cdot (D(p') - D(p))$$

$$= (x' - x) \cdot (d(x') - d(x))$$

$$\leq 0,$$

since $x, x' \geq 0$ under Assumption 1 and $d$ is monotonically decreasing on $R_+^N$. Thus we are done.

Note that if $d$ is linear, $d$ is monotonically decreasing if $A$ is positive semidefinite.

At this point, we have concluded our discussion of Theme (T1). We now move on to the second theme (T2), as follows.

4 Pricing models

Demand functions are basic ingredients in many economic decision making models, in particular, pricing models. In this section, we want to present some basic properties of game-theoretic pricing models which incorporate the piecewise smooth demand functions we defined in the preceding section. Such pricing models are represented by optimization (best response) problems with complementarity constraints. We will also discuss the conditions under which the complementarity constraints can be eliminated from these models.

A general pricing model can be represented as follows, in which (1) the objective of the seller is to maximize his profits; (2) for each product, the amount produced (nonnegative) is constrained by the corresponding market demand; (3) there may be additional constraints on the prices and quantities of products to be produced.
We consider a market (oligopolistic) of $M$ sellers, where given other sellers’ prices fixed, seller $i$’s best response pricing problem is

$$\max_{(p^i,q^i)} q^i^T p^i - c_i(q^i)$$

s.t.  
$$0 \leq q^i \leq D^i(p^i, p^{-i})$$
$$q^i \in Q_i$$
$$p^i \in G_i(p^{-i}) \subseteq R^N_i,$$

where $p^i \in R^{N_i}$ is the subvector of prices corresponding to the $N_i$ products offered by seller $i$, $p^{-i}$ is the subvector consisting of all subvectors $p^j$ except $p^i$, and the cost function $c_i : R^N_i \to R$ is assumed to be continuous throughout this section. Here $q^i \in R^N_i$ is the vector representing the quantities of the various products to be produced by seller $i$, $q^i \in Q_i$ represents any additional constraints on the quantities to be produced/sold, and $D^i$ is the demand for the corresponding products. Furthermore, $p^i \in G_i(p^{-i})$ represents the constraints on prices to be set by company $i$, when the prices of other companies, $p^{-i}$ are given. Prices are nonnegative, thus $p^i \in R^N_i$ is required. We denote $N = N_1 + \ldots + N_M$.

Note that in the pricing model, although company $i$’s strategy should be $(p^i, q^i)$, her best response problem is influenced by other companies only through prices $p^{-i}$. Thus, we will say that $p^i$ is a strategy of company $i$. When we say $p^i$ solves (16), we mean that there exists $q^i$ such that $(p^i, q^i)$ solves (16).

The $M$ companies in the market solve their best response problems simultaneously. The pricing policies obtained will be an example of a Generalized Nash Equilibrium (GNE). Recall that by a Nash Equilibrium $p^* = (p_1^*, p_2^*, \ldots, p_M^*)$, we mean that for each seller $i \in \{1, \ldots, M\}$, given $p^{-i}$, $p^i$ solves his best response problem (see Nash (1950)). In our case, each seller’s strategy set (set of possible prices) depends on other sellers’ strategies (prices); hence $p^*$ is called a Generalized Nash Equilibrium (as discussed in Harker (1991) and Pang (2002)).

Based on Definition 3 of the demand function $D$, problem (16) can be written as follows:

$$\max_{(p^i,x[i],q^i)} q^i^T p^i - c_i(q^i)$$

s.t.  
$$0 \leq q^i \leq d^i(x[i])$$
$$0 \leq d(x[i]) \perp p - x[i] \geq 0$$
$$q^i \in Q_i$$
$$p^i \in G_i(p^{-i}) \subseteq R^N_i,$$

where the vector $x[i] \in R^N$ consists of the corresponding variables in NCP($p$) and $[i]$ indicates that the vector is for the best response problem of seller $i$.

Note that $x[i]$ and $p$ are of $N$-dimension while $p^i$ and $q^i$ are of $N_i$-dimension. As before, when we say $p^*i$ solves (17), we mean that there exists $q^*i$ and $x[i]^*$ such that $(p^*i, x[i]^*, q^*i)$
solves (17). We also notice that under Assumption 1, if \( p^* = (p_1^*, p_2^*, \ldots, p_M^*) \) is an equilibrium and \((p^*, x[i]^*, q[i]^*)\), \( i = 1, \ldots, M \), are corresponding solutions of the best response problems, then \( x[1]^* = \ldots = x[M]^* \). This is because \( x[i]^* \) solves the NCP

\[
0 \leq d(x) \perp p^* - x \geq 0
\]

for the same \( p^* \) and the NCP has a unique solution.

The model (17) is an NCP constrained optimization problem. Readers can refer to Luo, Pang and Ralph (1996) for general properties of such problems and methods to solve them.

In accordance with our Theme (T2), we now investigate the structure of the pricing problem (17).

The following simple result is useful for showing later that the problem (17) is feasible and bounded.

**Lemma 13** For any \( p \in \mathbb{R}^N_+ \) and \( 0 \leq q_i \leq d_i(B(p)) \), we have \( q_i^T B^i(p) = q_i^T p^i \). Here \( B(p) \) is the solution of NCP\((p)\).

**Proof.** For each \( j \), if \( p_j > B_j(p) \), then by NCP\((p)\), \( d_j(B(p)) = 0 \). Hence \( q_j = 0 \) and \( q_j B_j(p) = q_j p_j = 0 \). Therefore, \( q_i^T B^i(p) = q_i^T p^i \) holds true. \( \square \)

**Proposition 14** Suppose Assumption 1 is satisfied, \( 0 \in Q_i \), and \( G_i(p^{-i}) \) is nonempty. Then the best response problem (17) is feasible, i.e., it has feasible solutions. If \( \Omega \) is bounded\(^4\), then the objective value of (17) is bounded from above on the feasible region.

**Proof.** Obviously, for any nonnegative \( p^i \in G_i(p^{-i}) \), \( (p^i, B(p), 0) \) is a feasible solution.

By Lemma 13, we can see that for any \((p^i, x[i]^i, q_i^i)\) feasible to (17), we will always have \( q_i^T p^i = q_i^T x[i]^i \). (As throughout our paper, superscript labels vectors, hence \( x[i]^i \) is the subvector of \( x[i] \) corresponding to the products of seller \( i \)). Since \( x[i] \in \Omega \) and \( \Omega \) is bounded, thus \( x[i]^i \) and \( q_i^i \) are both bounded \((0 \leq q_i^i \leq d_i(x[i]^i)) \) and in turn the objective value is bounded. \( \square \)

However, the set of optimal solutions of (17) can either be empty or have multiple solutions, because the feasible region need not be closed and convex. To illustrate this point, let us consider the simple case where \( d \) is linear and \( M = 1 \). For convenience, we omit the index for company. Let us denote

\[
B(G) = \{x \mid x = B(p), p \in G \subseteq \mathbb{R}^N_+\},
\]

\(^4\)If \( d \) is linear, then \( \Omega \) is bounded under Assumption 1 (see Lemma 5).
and rewrite the problem (17), by virtue of Lemma 13, as follows:

$$\max_{(x,q)} q^T x - c(q)$$

s.t. $0 \leq q \leq b - Ax$

$q \in Q$

$x \in B(G)$

The set $B(G)$ is the image of the set $G$ under the mapping $B$. Usually, such a set is nonconvex even if $G$ is convex. Thus the above program is in general nonconvex. Moreover, $B(G)$ need not be closed even if $G$ is closed, thus although the above problem may be bounded, it may not achieve an optimal solution.

**Example 4** Figure 2 shows an example in which $\Omega$ and $G$ are two-dimensional sets. Here $B(G)$ consists of two line segments $U_1U_2$ and $U_2U_3$, excluding the point $U_3$. Thus, $B(G)$ is neither convex nor closed.

![Figure 2. $B(G)$ is neither convex nor closed.](image)

Now, note that even if the constraints $q^i \in Q_i$ and $p^i \in G(p^{-i})$ are absent, the problem (17) is still nonconvex in general. We illustrate this here with a simple example.

**Example 5** Consider a simple pricing problem where a single seller offers 2 products, $d$ is linear, $c(q) = 0$ and $G = Q = R_+^2$ (omitting the index of company for convenience).
Since \( c(q) = 0 \) and \( Q = R^2_+ \), \( q = b - Ax = D \) at optimality. Thus, the problem (17) can be written as

\[
\begin{align*}
\max & \quad f(p) := p^T D(p) \\
\text{s.t.} & \quad p \in R^2_+.
\end{align*}
\] (18)

In the above figure, we see that \( D_1(\bar{p}) = d_1(B(\bar{p})) = 0 \), \( D_2(\bar{p}) = d_2(B(\bar{p})) > 0 \), \( D_1(\hat{p}) = d_1(B(\hat{p})) > 0 \), \( D_2(\hat{p}) = d_2(B(\hat{p})) = 0 \), and \( D_1(\tilde{p}) = D_2(\tilde{p}) = 0 \).

Thus, at the prices \( \tilde{p}, \bar{p} \) and \( \hat{p} \), we have the objective values \( f(\tilde{p}) = \tilde{p}_1 D_1(\tilde{p}) + \tilde{p}_2 D_2(\tilde{p}) = 0 \), \( f(\bar{p}) = \bar{p}_2 D_2(\bar{p}) > 0 \) and \( f(\hat{p}) = \hat{p}_1 D_1(\hat{p}) > 0 \) respectively. Since \( \tilde{p} = \lambda \bar{p} + (1 - \lambda)\hat{p} \) for some \( \lambda \in (0, 1) \), but \( f(\tilde{p}) < \lambda f(\bar{p}) + (1 - \lambda) f(\hat{p}) \), it is clear that the objective function of problem (18) is not concave, that is, problem (18) is nonconvex.

Having explained the structure of the pricing model (17), we wish to pursue Theme (T2) by considering possible simplifications of the problem (17).

It is not easy to solve problem (17) because it involves NCP constraints. In what follows, we will discuss some cases in which the complementarity constraints can be elimi-
inated. More precisely, we wish to reduce the problem (17) to the following problem:

$$\max_{(p', q') \in \Omega} \quad q^T p^i - c_i(q^i)$$

s.t. \hspace{1cm} 0 \leq q^i \leq \bar{d}(p) \quad \text{(19)}$

$$p^i \in Q_i$$

$$p^i \in G_i(p^{-i}) \subseteq R^N_i.$$

Such a simplification is not always possible. Here is a more technically complete version of the first example mentioned in section 2. Suppose \(d_1(p) = 20 - 3p_1 + 2p_2, \)
\(d_2(p) = 100 + p_1 - 4p_2, \)
\(c(q) = 0, \)
\(Q = \{q \mid q_1 + q_2 \leq 30\} \)
and \(G = \{p \geq 0 \mid p_1 \geq p_2\}.$$

For problem (19), we recall that the objective value of (19) is bounded from above by 600. As for problem (17), we can easily verify that \((p_1, p_2, x_1, x_2, q_1, q_2) = (k, 23, 22, 23, 0, 30)\)
for any \(k \geq 23\) is a feasible solution. The objective value of (17) at this point is 690, as we have obtained in section 2.

This shows that the optimal profit obtained using the pricing model (17) is higher than that obtained using the simplified model (19). Therefore, it is wrong to simplify (17) to (19) in this situation.

In situations like the above example, we need to consider prices outside \(\Omega\), that is, the NCP constraints inherited from the demand functions of multiple products are unavoidable. Failure to do so may lead to lower possible profits acquired by a seller, as shown above.

Now we wish to give a sufficient condition under which the game using best response model (17) can be reduced to the game using best response model (19). For convenience, we will say that the games (17) and (19) are the games involving the best response models (17) and (19) respectively. Note that where applicable, \(A_{ik}\) denotes the submatrix of \(A\) consisting of rows corresponding to seller \(i\)'s products and columns corresponding to seller \(k\)'s products.

**Theorem 15** Suppose that Assumption 1 is satisfied and \(d^i(p^i, p^{-i}) \leq d^i(p^i, \bar{p}^{-i})\) for any \((p^i, p^{-i}), (p^i, \bar{p}^{-i}) \in \Omega\) with \(p^{-i} \leq \bar{p}^{-i}\. Then under the hypothesis that for any \(p = (p^i, p^{-i}) \in R^N_i, p^i \in G_i(p^{-i})\) implies \(B^i(p) \in G_i(p^{-i})\) for all \(i = 1, \ldots, M\), if \(p^*\) is a GNE of the game (19), then \(p^*\) is a GNE of the game (17).

**Proof.** Let \(p^*\) be an equilibrium of (19). Then \(0 \leq q^{i*} \leq d^i(p^*)\) for \(i = 1, \ldots, M\). Thus, \(p^* \in \Omega\) and \((p^{i*}, p^{*}, q^{i*})\) is a feasible solution to (17) given \(p^{-i} = p^{-i*}\).

Let \((p^i, x[i], q^i)\) be a feasible solution of (17) given \(p^{-i} = p^{-i*}\). Since \(x[i]\) is the solution of the NCP corresponding to \(p = (p^i, p^{-i*})\), we have \(x[i]^{-i} \leq p^{-i*}\) and thus

$$0 \leq q^i \leq d^i(x[i]^i, x[i]^{-i}) \leq d^i(x[i]^i, p^{-i*}). \quad (20)$$

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Now because \( x[i] = B(p^i, p^{-i}) \), by the hypothesis in the theorem, \( p^i \in G_i(p^{-i}) \) implies \( x[i] \in G_i(p^{-i}) \). Thus, \( (x[i], q^i) \) is feasible to (19) given \( p^{-i} = p^{-is} \), where \( (p^*, q^*) \) is the optimal solution. Therefore,

\[
q^i \mathbf{x}[i] - c_i(q^i) \leq q^{is} \mathbf{p}^{is} - c_i(q^{is}).
\]

By Lemma 13, \( q^i \mathbf{p}^i = q^i \mathbf{x}[i] \). Thus we have

\[
q^i \mathbf{p}^i - c_i(q^i) \leq q^{is} \mathbf{p}^{is} - c_i(q^{is}).
\]

This shows that \((p^{is}, p^*, q^{is})\) is an optimal solution of (17) given \( p^{-i} = p^{-is} \).

Therefore \( p^* \) is an equilibrium of the game defined by the best response problem (17).

The condition \( d^i(p^i, p^{-i}) \leq d^i(p^i, \tilde{p}^{-i}) \) for any \((p^i, p^{-i}), (p^i, \tilde{p}^{-i}) \in \Omega \) with \( p^{-i} \leq \tilde{p}^{-i} \), implies that seller \( i \)'s products are substitutable for seller \( k \)'s products. For the linear case, this translates into \( A_{ik} \leq 0 \) \( \forall \ i, k, k \neq i \). However, note that the products sold by a seller can either be substitutes or complements for each other, because they are not restricted by the above condition.

Now if the constraint set \( G_i(p^{-i}) = R_{N_i}^+ \) in (17), the hypothesis that \( p^i \in G_i(p^{-i}) \) implies \( B^i(p) \in G_i(p^{-i}) \) is satisfied under Assumption 1. In the papers Bernstein and Federgruen (2004), Dai, Chao, Fang and Nuttle (2004), Gallego and Van Ryzin (1997), Garcia-Gallego and Georgantzis (2001), Maglaras and Meissner (2003), and Roy, Hanssens and Raju (1994), either no additional constraints on \( p^i \) are explicitly mentioned, i.e. \( G_i(p^{-i}) = R_{N_i}^+ \), or \( p^i \) is constrained to lie in some interval \([p_{min}, p_{max}]\). Note that if \( p_{min} = 0 \), it is easy to see that the hypothesis in Theorem 15 will be satisfied. Thus the simpler pricing model (19) can be used instead of model (17) in solving the game. In this case, Theorem 15 can be viewed as a rigorous justification for the validity of the pricing models discussed in their papers.

The reduction of the model (17) to the model (19) tremendously simplifies both computation and theoretical analysis. However in general, even if the hypothesis “for any \( p \in R_{N_i}^+ \), \( p^i \in G_i(p^{-i}) \) implies \( B^i(p) \in G_i(p^{-i}) \) for all \( i = 1, \ldots, M^* \) is satisfied, it may not be possible to reduce the game (17) to the game (19) if seller \( i \)'s products are not assumed substitutable for seller \( k \)'s products, \( \forall i \neq k \). The following example shows that an equilibrium for the game (19) need not be an equilibrium for the game (17).

**Example 6** Suppose there are 2 sellers in the market, where seller 1 produces products 1 and 2, \( q^1 = (q_1; q_2) \), and seller 2 produces product 3, \( q^2 = q_3 \). Let \( d_1(p) = 20 - 2p_1 - p_3 \), \( d_2(p) = 13 - 2p_2 - p_3 \) and \( d_3(p) = 10 - 1.5p_2 - 2p_3 \). The cost functions facing the sellers are \( c_1(q^{i}) = 0q_1 + 5q_2 \) and \( c_2(q^{2}) = 0q_3 \). In addition, let \( Q_1 = G_1 = R_{N_1}^2 \) and \( Q_2 = G_2 = R_{N_2} \). It is easy to show that the matrix \( A \) is a P-matrix and the hypothesis in Theorem 15 is satisfied.
We obtained $p^1 = (4.91, 5.66)$, $p^2 = 0.38$ to be a Nash Equilibrium of the game (19), with corresponding $q^{1*} = (9.8, 1.3)$, $q^{2*} = 0.75$. At this equilibrium, the profits for seller 1 and seller 2 are 48.98 and 0.29 respectively.

Now, we check if the above equilibrium is also an equilibrium for the game (17). Let us look at the best response problem (17) for seller 1, given $p_2^* = p_3^* = 0.38$ fixed. Consider the vectors $\bar{p}^1 = (\bar{p}_1, \bar{p}_2) = (4.95, 6.4)$, $\bar{q}^1 = (\bar{q}_1, \bar{q}_2) = (9.9, 0)$ and $\bar{x}[1] = (4.95, 6.4, 0.2)$. It is easy to check that all the constraints of (17) for seller 1 are satisfied. At this feasible solution, we find that the profit seller 1 obtains is 49.01, higher than 48.98 obtained at $p^1 = (4.91, 5.66)$. This shows that $p^1 = (4.91, 5.66)$ is not a best response to $p_2^* = 0.38$ for the game (17). Thus, $(p^1, p^2)$ is not a Nash Equilibrium of the game (17).

Hence, a GNE of the game (19) need not be a GNE of the game (17).

Let us briefly explain why we cannot simply reduce the game (17) to the game (19). Suppose $p^*$ is an equilibrium of the game (19). Then $(p^i, p^*, q^*)$ is a feasible solution to (17) given $p^i = p^i - p^*$. This means that given $p^i = p^i - p^*$, the optimal profit obtained by solving (17) cannot be lower than that obtained by solving (19). However, if $(\tilde{p}^i, \tilde{x}[i], \tilde{q}^i)$ is optimal to (17) given $p^i = p^i - p^*$, $\tilde{q}^i$ may not be feasible to (19) given $p^i = p^i - p^*$. As $\tilde{x}[i] - p^i$ is possible and $A_{ik} \not\subseteq 0$ for $k \neq i$ (seller i’s products may not be substitutable for seller k’s products), the inequality $\tilde{q}^i \leq b^i - A_{ii}\tilde{x}[i] - \sum_{k\neq i} A_{ik}p^k$ may not be satisfied. Thus, it is possible to obtain a higher maximum profit using model (17).

For the case where there is only one company ($M = 1$), we can present a more specific relationship between (17) and (19). For convenience, we omit the index for company, e.g., now $p$ represents the price vector of the company and we write $G$ instead of $G_i(p - i)$. We rewrite the models (17) and (19) for this case as follows:

$$\max_{(p, x, q)} q^T p - c(q)$$
\begin{align*}
\text{s.t.} & \quad 0 \leq q \leq d(x) \\
& \quad 0 \leq d(x) \perp p - x \geq 0 \\
& \quad q \in Q \\
& \quad p \in G \subseteq R_+^N,
\end{align*}

and

$$\max_{(p, q)} q^T p - c(q)$$
\begin{align*}
\text{s.t.} & \quad 0 \leq q \leq d(p) \\
& \quad q \in Q \\
& \quad p \in G \subseteq R_+^N.
\end{align*}

**Theorem 16** Under Assumption 1, if for any $p \in R_+^N$, $p \in G$ implies $B(p) \in G$, then the models (21) and (22) are ‘equivalent’ in the following sense:
(i) If $p^*$ is an optimal solution of the model (21), then $B(p^*)$ is an optimal solution of the model (22).

(ii) If $p^*$ is an optimal solution of the model (22), then any $p$ satisfying $B(p) = p^*$ and $p \in G$ is an optimal solution of (21).

(iii) The models (21) and (22) have the same optimal objective values.

Proof. Let $v_1$ and $v_2$ be the optimal objective values of (21) and (22) respectively.

Suppose that $(p^*, x^*, q^*)$ is optimal to (21). Then $x^* = B(p^*)$. Under the hypothesis in the theorem, $p^* \in G$ implies $x^* \in G$. Thus, $(p, q) = (x^*, q^*)$ is feasible to (22).

By Lemma 13,

$$v_1 = q^*T p^* - c(q^*) = q^*T x^* - c(q^*) \leq v_2. \quad (23)$$

If $(p^*, q^*)$ is optimal to (22), then for any $p$ satisfying $B(p) = p^*$ and $p \in G$, $p^*$ is the solution of NCP($p$), i.e., $0 \leq d(p^*) \perp p - p^* \geq 0$. Hence it is easy to see that $(p, p^*, q^*)$ is feasible to (21). By Lemma 13,

$$v_2 = q^*T p^* - c(q^*) = q^*T p - c(q^*) \leq v_1. \quad (24)$$

Combining (23) and (24), we have $v_1 = v_2$ and (iii) is proven. Since the equality holds in (23) and (24), $(x^*, q^*)$ is optimal to problem (22) and $(p, p^*, q^*)$ is optimal to problem (21). Thus we have proven (i) and (ii).

Note that in the above theorem, there are no specific conditions on the relationship between the demand for product $i$ and the price of product $j$ ($j \neq i$). Hence, the products may be substitutes or complements for each other.

It is interesting to see that Theorem 16 can allow us to reduce a nonconvex problem (21) (as shown in Example 5) to a convex problem (22) (if $d$ is concave, $c(q)$ is a convex function and $G$ and $Q$ are convex sets).

We have shown with examples that some commonly used pricing models may lead to inferior results. On the other hand, under certain conditions, these simpler models are essentially “equivalent” to the NCP constrained models. Indeed, it may be possible to uncover other tractable models via simplifications of the NCP constrained model. However, we wish to emphasize that the use of any simplified model requires rigorous justifications, and such justifications are now made possible by the virtue of the fundamental and complete NCP constrained model.
5 Conclusion

In this paper, we have formally constructed a model of piecewise smooth demand functions for a market of multiple products, using a nonlinear complementarity problem (NCP). Based on this, we introduced an NCP constrained best response pricing problem for each seller involved in a pricing game. Some properties of our demand and pricing models were presented. In particular, we have shown that our demand function is monotone, which is an expression of the law of demand for a market of multiple products. Under certain conditions, we have also shown that the complementarity constrained pricing model can be simplified by eliminating the complementarity constraints. Such a simplification is also possible for many other optimization problems and games involving our model of demand functions. However, on the other hand, there are also many instances in which such simplifications can lead to wrong models, as shown by some of the examples we have presented. In these situations, the NCP constraints inherited from the demand functions are unavoidable and remain as a core structure in the models. We wish to emphasize that a complete demand function on the entire set of nonnegative prices is necessary, though it may appear unduly complex. It is a cornerstone for complete pricing models and many other models and is the origin and foundation for any simplified models.

Some future research directions:

- The theoretical analysis, e.g., existence and uniqueness, of Nash Equilibria for games involving our pricing models with complementarity constraints.

- The search for new ways in which NCP constrained decision-making models may be simplified to forms which yield more insight and are more tractable.

- The study of stochastic demand models. Our deterministic demand model can be the backbone of many realistic, stochastic demand models.

- The investigation of applications of demand functions is a rich area for researchers and practitioners. The pricing models and their properties studied in this paper are merely simple examples in this direction.

- As a special example of an MPEC, cf. Luo, Pang and Ralph (1996), the NCP constrained pricing model presented in this paper may be solved by methods specifically tailored for it. New theory and algorithms that exploit the special structure of the model may be developed.

References


