Nonminimal Product Differentiation as a Bargaining Outcome
(Supplement)

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This supplement provides some details omitted in the body of the paper.
1 Outline

Some details and proofs of results omitted in the body of the paper is provided in this supplement. The presentation corresponds to the order of appearance in the body of the paper. The contents of each section are as follows.

Section 2: It is claimed on pages 4 and 5 of the paper that when the firms charge the Nash bargaining solution prices, (P1)–(P4) are satisfied. This is verified here.

In the context of Figure 3, it is claimed on page 10 that the Nash bargaining solution outcome lies below the line where the profits are shared in proportion to one-shot Nash equilibrium profits. This is also proved in this section.

Section 3: When the firms share profits on the PPF in proportion to one-shot Nash equilibrium profits, (P2) and (P4) are satisfied (page 5 of the paper). This is verified here.

Section 4: It is claimed on page 7 of the paper that \( \bar{M} = 1 + t(x_1 + x_2 - 1)\bar{K} \). This is proved in this section.

Section 5: Two equivalent equations which characterize the Kalai–Smorodinski solution are given. On page 9 of the paper, after Theorem 2 is stated some of the properties of (2) such as \( x^*_1 \) is unique and lies between 0.35 and 0.5. \( x^*_1 \) tends to 0.35 as \( A/t \) tends to infinity etc. are noted. These are established here.

Section 6: This section notes some of the properties of the Kalai–Smorodinski solution. In particular, it is shown that it satisfies (P1)–(P4).

Section 7: Some further properties of the Kalai-Smorodinski solution are noted in this section.

The behavior of the ratio of profits as captured by \( M \) is examined. Some properties of \( \bar{M} \) and \( \bar{K} \) are noted.

Section 8: This section provides an important inequality at symmetric locations. It shows that the sign of \( K - \bar{K} \) is determined by the LHS of (2) in Theorem 2.

Section 9: Theorem 2 is proved in this section.

Section 10: This section provides some details regarding the convexity of the set of equilibria.

Section 11: It is claimed on page 11 of the paper that point L lies above point V and below line EG in Figure 3. It is claimed on page 13 of the paper that the functions \( (4 - x_1 - x_2)^2\bar{K} \) and \( h \) are increasing functions of \( x_1 \). These are proved in this section.
2 The Nash Bargaining Solution

Jehiel (1992) has examined the implications of the Nash bargaining solution prices and has shown that if the firms choose locations in the initial period and in the subsequent infinitely many periods charge the Nash bargaining solution prices then there is a unique equilibrium in which the firms locate at the center of the market. The following arguments are somewhat different from his. It is also shown that when the firms charge the Nash bargaining solution prices, (P1)-(P4) are satisfied.

The Nash bargaining solution is determined by maximizing the product \( N = (\Gamma_1 - \Gamma_{1N})(\Gamma_2 - \Gamma_{2N}) \). When \( x_1 < x_2 \), the derivatives with respect to the prices are given by

\[
2t(x_2 - x_1) \frac{\partial N}{\partial p_1} = (\Gamma_1 - \Gamma_{1N})p_2 + (\Gamma_2 - \Gamma_{2N})[2t(x_2 - x_1)z - p_1] \\
2t(x_2 - x_1) \frac{\partial N}{\partial p_2} = (\Gamma_1 - \Gamma_{1N})[2t(x_2 - x_1)(1 - z) - p_2] + (\Gamma_2 - \Gamma_{2N})p_1
\]

The second derivatives are negative.

Since \( \partial N/\partial p_1 \geq 0 \) and \( \partial N/\partial p_2 \geq 0 \),

\[
\frac{p_1 - 2t(x_2 - x_1)z}{p_2} \leq \frac{\Gamma_1 - \Gamma_{1N}}{\Gamma_2 - \Gamma_{2N}} \leq \frac{p_1}{p_2 - 2t(x_2 - x_1)(1 - z)}
\]

Let \( x_1 \to x_2 \). Then \( p_1 \to p_2 \) and \( \Gamma_{1N} \) and \( \Gamma_{2N} \) tend to zero. So, \( \lim_{x_1 \to x_2} (\Gamma_1/\Gamma_2) = 1 \). Thus the profits are shared equally if the firms are located at the same point. So, (P3) is satisfied.

Clearly at symmetric locations, \( x_1 + x_2 = 1 \), the profits of the two firms are identical. So, (P2) is satisfied. The following two claims provide essential ingredients for the characterization of equilibrium. The first specifies the prices in different situations. The second provides a bound for market share. Together they ensure that (P4) is satisfied.

**Lemma 2** Let \( x_1 < x_2 \). If \( x_1 + x_2 - 1 > 0 \) then either \( p_1 = A - tx_1^2 \) or, \( p_1 = A - t(z - x_1)^2 \) for some \( z \) in \((x_1, x_2)\). If \( x_1 + x_2 - 1 < 0 \) then either \( p_2 = A - t(1 - x_2)^2 \) or, \( p_2 = A - t(z - x_2)^2 \) for some \( z \) in \((x_1, x_2)\).

**Proof** Let \( x_1 + x_2 - 1 > 0 \) and suppose to the contrary that no consumer of Firm 1 pays the reservation price. So, it must be the case that \( p_2 = A - t(1 - x_2)^2 \).

Therefore, \( p_2 = A - t(1 - x_2)^2 > A - tx_1^2 > p_1 \). Since \( p_2 > p_1 \), \( \Gamma_1/\Gamma_2 > (x_1 + x_2)/(2 - x_1 - x_2) \). The latter is greater than \( \Gamma_{1N}/\Gamma_{2N} \) since \( x_1 + x_2 - 1 > 0 \). Therefore, \((\Gamma_1 - \Gamma_{1N})/(\Gamma_2 - \Gamma_{2N})\)
> $$\Gamma_1N/\Gamma_2N > 1$$. It now follows that $$\partial N/\partial p_1$$ is positive. So, $$N$$ can be increased by increasing $$p_1$$ and yet the entire market will be served. This is a contradiction.

**Lemma 3** Let $$x_1 < x_2$$. If $$x_1 + x_2 - 1 > 0$$ then $$z > 1/2$$. If $$x_1 + x_2 - 1 < 0$$ then $$z < 1/2$$.

**Proof** Let $$x_1 < x_2$$ and $$x_1 + x_2 - 1 > 0$$. Suppose that $$z \leq 1/2$$. Then $$p_2 < p_1$$.

$$\Gamma_1 = p_1z$$ and $$\Gamma_2 = p_2(1 - z) = [p_1 + t(x_2 - x_1)](2z - x_1 - x_2)(1 - z)$$. By the preceding Lemma, $$p_1 = A - tx_1^2$$ or, $$p_1 = A - t(z - x_1)^2$$. In either case, $$\Gamma_1$$ is increasing and $$\Gamma_2$$ is decreasing in $$z$$. Therefore, $$(\Gamma_1 - \Gamma_1N)/(\Gamma_2 - \Gamma_2N)$$ is an increasing function of $$z$$. When $$z = 1/2$$, $$(\Gamma_1 - \Gamma_1N)/(\Gamma_2 - \Gamma_2N) < 1$$. Hence, $$(\Gamma_1 - \Gamma_1N)/(\Gamma_2 - \Gamma_2N) < 1$$ if $$z \leq 1/2$$.

It follows that $$\partial N/\partial p_2 > 0$$. Therefore, some customer of Firm 2 must pay the reservation price. $$p_2 = A - t(1 - x_2)^2$$ implies $$p_2 > p_1$$, a contradiction. So, $$p_2 = A - t(z - x_2)^2$$ and therefore, $$p_1 = A - t(z - x_1)^2$$.

$$N$$ can be viewed as a function of $$z$$ and $$\partial N/\partial z$$ is given by

$$[A - t(z - x_1)^2 - 2t(z - x_1)z][\Gamma_2 - \Gamma_2N] - [A - t(z - x_2)^2 + 2t(z - x_2)(1 - z)][\Gamma_1 - \Gamma_1N]$$

This is positive since $$\Gamma_2 - \Gamma_2N > \Gamma_1 - \Gamma_1N$$ and $$A - t(z - x_1)^2 - 2t(z - x_1)z > A - t(z - x_2)^2 + 2t(z - x_2)(1 - z)$$.

So, $$N$$ can be increased and is not maximized. Hence, if $$x_1 + x_2 - 1 > 0$$ then $$z > 1/2$$.

So, both firms locating at the center is an equilibrium. The only other candidates for equilibrium are the symmetric ones $$(x_1, x_2)$$ with $$x_1 \geq 1/4$$.

Some further arguments are required to rule out the symmetric locations $$(x_1, x_2)$$ with $$1/4 \leq x_1 < 1/2$$ as candidates for equilibrium.

Consider a pair of symmetric locations $$(x_1, x_2)$$ with $$1/2 < x_2 \leq 3/4$$ and let there be a small increase in the location of Firm 1 to $$x_1 + \epsilon$$. By Lemma 2, $$p_1 = A - t(x_1 + \epsilon)^2$$. When $$p_2 = p_1$$, the entire market is served and no customer of Firm 2 pays the reservation price. In this case $$2t(x_2 - x_1 - \epsilon)(\partial N/\partial p_2)$$ is

$$t(x_2 - x_1 - \epsilon)(2 - x_1 - x_2 - \epsilon)(\Gamma_1 - \Gamma_1N) + \left[\frac{2t(x_2 - x_1 - \epsilon)}{3} - p_1\right](x_1 + x_2 + \epsilon - 1)p_1$$

This is positive when $$\epsilon$$ is zero and hence positive for $$\epsilon$$ small.

Therefore, $$p_2 > p_1$$ at locations $$(x_1 + \epsilon, x_2)$$ and the market share of Firm 1 is greater than $$(x_1 + x_2 + \epsilon)/2 = (1 + \epsilon)/2$$. In order to prove that Firm 1 gains by changing its location
it suffices to show that \([A - t(x_1 + \epsilon)^2](1 + \epsilon)/2 > [A - tx_2^2]/2\). This holds since the expression is increasing in \(\epsilon\).

This shows that if the firms charge the Nash bargaining prices then the unique equilibrium is when both firms locate at the center of the market.

Note that \(p_2 > p_1\) can hold at locations \((x_1 + \epsilon, x_2)\) if \(\epsilon\) is small. In that case, \(\Gamma_1/\Gamma_2 > (x_1 + x_2 + \epsilon)/(2 - x_1 - x_2 - \epsilon) > \Gamma_{1N}/\Gamma_{2N}\).

Therefore, \(\Gamma_1/\Gamma_2 > \Gamma_{1N}/\Gamma_{2N}\) can hold if \(x_1 + x_2 - 1 > 0\). Geometrically, this means the Nash bargaining solution outcome lies below the line where the profits are shared in proportion to one-shot Nash equilibrium profits.
3 The Proportional Sharing Rule

Suppose that the firms share profits on the PPF in proportion to one-shot Nash equilibrium profits as in Friedman and Thisse (1993), i.e., $\Gamma_1/\Gamma_2 = \Gamma_{1N}/\Gamma_{2N}$. Then the conclusions of the preceding two Lemmas hold.

Let $x_1 + x_2 - 1 > 0$ and suppose that $p_2 = A - t(1 - x_2)^2$. Then $p_2 = A - t(1 - x_2)^2 > A - tx_1^2 > p_1$. Hence, $\Gamma_1/\Gamma_2 > (x_1 + x_2)/(2 - x_1 - x_2) > \Gamma_{1N}/\Gamma_{2N}$, a contradiction. Therefore, if $x_1 + x_2 - 1 > 0$ then $p_1 = A - tx_1^2$ or, $p_1 = A - t(z - x_1)^2$.

Suppose that $x_1 + x_2 - 1 > 0$ and $z \leq 1/2$. Then $(\Gamma_1 - \Gamma_{1N})/(\Gamma_2 - \Gamma_{2N}) < 1 < \Gamma_{1N}/\Gamma_{2N}$.

This is equivalent to $\Gamma_1/\Gamma_2 < \Gamma_{1N}/\Gamma_{2N}$, a contradiction. So, if $x_1 + x_2 - 1 > 0$ then $z > 1/2$.

This shows that (P4) holds under this sharing rule. Clearly, this sharing rule satisfies the symmetry condition (P2).

To show that central agglomeration is the unique equilibrium, consider a pair of symmetric locations $(x_1, x_2)$ with $x_2 \leq 3/4$. Let $\hat{x}_1 = x_1 + \epsilon$. Let $p_1$, $p_2$ and $z$ be the prices and market share of Firm 1 under the solution at locations $(x_1 + \epsilon, x_2)$. Then $p_1 = A - t\hat{x}_1^2$ and $p_2 = p_1 + t(x_2 - x_1)(2z - x_1 - x_2)$.

Let $\hat{\sigma} = (A - tx_1^2)/[2(A - t\hat{x}_1^2)]$ and $\hat{p}_2 = p_1 + t(x_2 - x_1)(2\hat{\sigma} - x_1 - x_2)$. Then $\Gamma_1(x_1 + \epsilon, x_2) = p_1 z$ and $\Gamma_1(x_1, x_2) = (A - tx_1^2)/2 = p_1 \hat{\sigma}$. Therefore, $\Gamma_1(x_1 + \epsilon, x_2) > \Gamma_1(x_1, x_2)$ iff $z > \hat{\sigma}$.

Since $p_1 = A - t\hat{x}_1^2$, $\Gamma_1/\Gamma_2$ is an increasing function of $z$. Therefore, $z > \hat{\sigma}$ iff $\Gamma_{1N}/\Gamma_{2N} > (p_1 \hat{\sigma})/[\hat{p}_2(1 - \hat{\sigma})]$. Since $\Gamma_{1N}/\Gamma_{2N} = (3 + \epsilon)^2/(3 - \epsilon)^2$, this can be written as

$$2(3 + \epsilon)^2 \hat{p}_2(1 - \hat{\sigma}) - (3 - \epsilon)^2(A - t\hat{x}_1^2) > 0$$

By substituting $\hat{x}_1 = x_1 + \epsilon$, this can be expressed as a function of $\epsilon$. It can be shown that the LHS is an increasing function of $\epsilon$. When $\epsilon = 0$, the LHS equals zero. Hence, the LHS is positive for $\epsilon > 0$.

This shows that central agglomeration is the unique equilibrium.
4 The Relationship between $\tilde{M}$ and $\tilde{K}$

$$\tilde{M} = \frac{A - t(1 - x_2)^2 - 2\Gamma_{1N}}{2[A - tx_1^2 + t(x_2 - x_1)(2\sigma - x_1 - x_2)](1 - \sigma) - 2\Gamma_{2N}}$$

$$2(1 - \sigma) = 1 - \frac{t(1 + x_1 - x_2)(x_1 + x_2 - 1)}{A - tx_1^2}$$

$$2\sigma - x_1 - x_2 = -(x_1 + x_2 - 1) \left[ 1 - \frac{t(1 + x_1 - x_2)}{A - tx_1^2} \right]$$

Note that

$$A - t(1 - x_2)^2 - 2[A - tx_1^2 + t(x_2 - x_1)(2\sigma - x_1 - x_2)](1 - \sigma)$$

$$= A - t(1 - x_2)^2 - [A - tx_1^2 + t(x_2 - x_1)(2\sigma - x_1 - x_2)] \left[ 1 - \frac{t(1 + x_1 - x_2)(x_1 + x_2 - 1)}{A - tx_1^2} \right]$$

$$= 2t(1 + x_1 - x_2)(x_1 + x_2 - 1) - t(x_2 - x_1)(2\sigma - x_1 - x_2)$$

$$+ t(x_2 - x_1)(2\sigma - x_1 - x_2) \frac{t(1 + x_1 - x_2)(x_1 + x_2 - 1)}{A - tx_1^2}$$

$$= 2t(1 + x_1 - x_2)(x_1 + x_2 - 1) + t(x_2 - x_1)(x_1 + x_2 - 1) \left[ 1 - \frac{t(1 + x_1 - x_2)}{A - tx_1^2} \right]$$

$$+ t(x_2 - x_1)(2\sigma - x_1 - x_2) \frac{t(1 + x_1 - x_2)(x_1 + x_2 - 1)}{A - tx_1^2}$$

$$= t(2 + x_1 - x_2)(x_1 + x_2 - 1) - t(x_2 - x_1)(x_1 + x_2 - 1) \frac{t(1 + x_1 - x_2)}{A - tx_1^2}$$

$$+ t(x_2 - x_1)(2\sigma - x_1 - x_2) \frac{t(1 + x_1 - x_2)(x_1 + x_2 - 1)}{A - tx_1^2}$$

$$= t(2 + x_1 - x_2)(x_1 + x_2 - 1) - t(x_2 - x_1)(x_1 + x_2 - 1)(x_1 + x_2 + 1 - 2\sigma) \frac{t(1 + x_1 - x_2)}{A - tx_1^2}$$

$$= t(2 + x_1 - x_2)(x_1 + x_2 - 1) - t(x_2 - x_1)(1 + x_1 - x_2)(x_1 + x_2 - 1)g$$

where

$$g = \frac{t(x_1 + x_2 + 1 - 2\sigma)}{A - tx_1^2}$$
So, the difference between the numerator and the denominator of \( \tilde{M} \) is

\[
t(x_1 + x_2 - 1)[2 + x_1 - x_2 - (x_2 - x_1)(1 + x_1 - x_2)g] - 2\Gamma_{1N} + 2\Gamma_{2N}
\]

\[
= t(x_1 + x_2 - 1)[2 + x_1 - x_2 - (x_2 - x_1)(1 + x_1 - x_2)g] - \frac{4}{3}t(x_2 - x_1)(x_1 + x_2 - 1)
\]

\[
= t(x_1 + x_2 - 1) \left[ 2 + x_1 - x_2 - (x_2 - x_1)(1 + x_1 - x_2)g - \frac{4}{3}(x_2 - x_1) \right]
\]

\[
= t(x_1 + x_2 - 1) \left[ 2 - (x_2 - x_1) \left( \frac{7}{3} + (1 + x_1 - x_2)g \right) \right]
\]

Therefore,

\[
\tilde{M} = 1 + t(x_1 + x_2 - 1) \frac{2 - (x_2 - x_1) \left( \frac{7}{3} + (1 + x_1 - x_2)g \right)}{2[A - tx_1^2 + t(x_2 - x_1)(2\sigma - x_1 - x_2)](1 - \sigma) - 2\Gamma_{2N}}
\]

which is the same thing as \( 1 + t(x_1 + x_2 - 1)\tilde{K} \).
5 The Kalai–Smorodinski Bargaining Solution

5.1 Mathematical Description of the Solution

Let $M_i$ denote the maximum profit of Firm $i$ such that the profit of the other firm is greater than or equal to its one-shot Nash equilibrium profit. In Figure 1, these points are respectively B and D. Point C has the coordinates $(M_1, M_2)$. Let $M = (M_1 - \Gamma_{1N})/(M_2 - \Gamma_{2N})$. Then the Kalai–Smorodinski solution is given by the pair of profits $(\Gamma_1, \Gamma_2)$ on the PPF such that

$$
\frac{\Gamma_1 - \Gamma_{1N}}{\Gamma_2 - \Gamma_{2N}} = \frac{M_1 - \Gamma_{1N}}{M_2 - \Gamma_{2N}}
$$

Equivalently, the ratio of profits is given by

$$
\frac{\Gamma_1}{\Gamma_2} = \frac{\Gamma_{1N}/\Gamma_2}{\Gamma_{2N}/\Gamma_2} (1 - \Gamma_{2N}/\Gamma_2) + M \left(1 - \frac{\Gamma_{2N}}{\Gamma_2}\right)
$$

5.2 Behavior of the Equation in Theorem 2

Since $A \geq 3t$, both \([A/t - x_1^2](14 - 40x_1^2) - (1 - 2x_1^2)(20 - 76x_1^2)\) and \([A/t - x_1^2](9 - 60x_1^2) + 18x_1^2(1 - 2x_1^2)\) are decreasing functions of $x_1^*$. So the LHS of (2) is decreasing in $x_1^*$.

It is easy to see that the LHS is negative when $x_1^* = 1/2$. When $x_1^* = 7/20$ the LHS is positive (shown below). So, $x_1^*$ is unique and lies strictly between 0.35 and 0.5.

When $A/t = 3$, $x_1^* = 0.366$ (approx). The derivative of the LHS with respect to $A/t$ is negative in a neighborhood of $x_1^*$. So, as $A/t$ increases the LHS decreases near $x_1^*$, i.e., $x_1^*$ has to decrease to maintain equality in (2). This means as $A/t$ increases more equilibria appear. In the extreme case, $x_1^*$ tends to 0.35 as $A/t$ tends to infinity.

(1) It is shown that the LHS of (2) is positive when $x_1^* = 7/20 = 0.35$. One needs to show that

$$
\left(\frac{A}{t} - x_1^{*2}\right)^2 (76x_1^{*2} - 20) + \left(\frac{A}{t} - x_1^{*2}\right) (1 - 2x_1^*)(9 - 60x_1^*) + 18x_1^*(1 - 2x_1^*)^2 > 0
$$

It suffices to show that

$$
\left(\frac{A}{t} - x_1^{*2}\right) (76x_1^* - 20) > 3(1 - 2x_1^*)(20x_1^* - 3)
$$

Since $x_1^* \geq 1/3$, $3(1 - 2x_1^*) \leq 1$. Therefore, it suffices to show that $2(76x_1^* - 20) > 20x_1^* - 3$. This is $132x_1^* > 37$. Since $x_1^* \geq 1/3$, $132x_1^* \geq 44$. So, the inequality holds.

(2) The derivative of the LHS of (2) with respect to $A/t$ is examined below. The
derivative of the LHS of (2) with respect to $A/t$ is

$$3 \left( \frac{A}{t} - x_1^* \right)^2 (14 - 40x_1) - 2 \left( \frac{A}{t} - x_1^* \right) (1 - 2x_1)(20 - 76x_1) + (1 - 2x_1)^2(9 - 60x_1)$$

At $x_1^*$, this is

$$\left( \frac{A}{t} - x_1^* \right)^2 (1 - 2x_1^*)(20 - 76x_1^*) - 2(1 - 2x_1^*)^2(9 - 60x_1^*) - \frac{54x_1^*(1 - 2x_1)^3}{(A/t) - x_1^*}$$

To show that this is negative it suffices to show that $[(A/t) - x_1^*]((10 - 38x_1^*) - (1 - 2x_1^*)(9 - 60x_1^*)$ is negative. This is equivalent to $3(1 - 2x_1^*)(20x_1^* - 3) < [(A/t) - x_1^*]((38x_1^* - 10)$.

It suffices to show that $20x_1^* - 3 < 2(38x_1^* - 10)$ or, $17 < 56x_1^*$. This holds, for $x_1^* \geq 1/3$.

This shows that the derivative of the LHS of (2) with respect to $A/t$ is negative in a neighborhood of $x_1^*$.

(3) $x_1^*$ tends to 0.35 as $A/t$ tends to infinity. Since $1/3 \leq x_1^* \leq 1/2$, $18x_1^*(1 - 2x_1^*)^2 \leq 1$.

On the other hand, $(A/t) - x_1^* \geq 2$ and $60x_1^* - 9 \geq 11$. So $[(A/t) - x_1^*]((60x_1^* - 9) - 18x_1^*(1 - 2x_1^*)^2 > 0$. Hence,

$$\left( \frac{A}{t} - x_1^* \right)^2 (14 - 40x_1^*) > (1 - 2x_1^*)(20 - 76x_1^*)$$

$$\left( \frac{A}{t} - x_1^* \right)^2 (40x_1^* - 14) < (1 - 2x_1^*)(76x_1^* - 20)$$

$$\frac{A}{t} < x_1^* + \frac{(1 - 2x_1^*)(76x_1^* - 20)}{40x_1^* - 14}$$

So as $A/t$ tends to infinity $40x_1^* - 14$ tends to zero or $x_1^*$ tends to 0.35.
6 Towards Characterizing the Kalai–Smorodinski Solution

6.1 General Properties

The structure of equation (2) and the consequent continuum of equilibria suggest the intricate nature of the Kalai–Smorodinski bargaining solution. The details are given below and in the next section. Theorem 2 is proved in section 9.

The first step requires the determination of the maximal profit of a firm \((M_1 \text{ or } M_2)\) subject to the constraint that the profit of the other firm be greater than or equal to its Nash equilibrium profit. Consider the problem of maximizing the profit of Firm 1 subject to the constraint that the profit of Firm 2 is greater than or equal to \(\Gamma_2N\). Suppose at such a solution the profit of Firm 2 is strictly greater than \(\Gamma_2N\). It must be the case that the entire market is being served and at least one consumer is paying the reservation price. So the price of each firm is greater than or equal to \(2t\). If the price of Firm 1 is reduced slightly, keeping the price of Firm 2 constant, then the profit of Firm 1 increases and the profit of Firm 2 still remains above \(\Gamma_2N\). This contradiction shows that if the profit of Firm 1 is maximized subject to the constraint that the profit of Firm 2 is greater than or equal to \(\Gamma_2N\) then the profit of Firm 2 is \(\Gamma_2N\). In other words, \(M_1\) corresponds to a profit of \(\Gamma_2N\) for Firm 2. Similarly, \(M_2\) corresponds to a profit of \(\Gamma_1N\) for Firm 1. Geometrically, these are the points B and D in Figure 1.

To determine the solution, one needs to work with three sets of prices: (i) the prices which determine \(M_1\), (ii) the prices which determine \(M_2\) and (iii) the prices which determine the actual profits \(\Gamma_1\) and \(\Gamma_2\). These prices are denoted respectively by \((v_1, v_2)\), \((w_1, w_2)\) and \((p_1, p_2)\). The market share of Firm 1 in these three cases are denoted by \(\alpha\), \(\beta\) and \(z\) respectively.

The next step deals with the limit behavior of the ratio of profits as one firm moves arbitrarily close to the other. Without loss of generality let \(x_2 \geq 1/2\). As \(x_1\) tends to \(x_2\) both \(\Gamma_1N\) and \(\Gamma_2N\) tend to zero. Therefore, \(M_1\) and \(M_2\) in the limit correspond to each of the firms in turn having the entire market. So, as \(x_1\) approaches \(x_2\) both \(M_1\) and \(M_2\) approach \(A - tx^2\), i.e., \(M\) tends to 1. So from (3), \(\lim_{x_1 \rightarrow x_2} (\Gamma_1/\Gamma_2) = 1\). So the profits are shared equally if the firms are located at the same point. Thus, the Kalai–Smorodinski solution satisfies (P3).

Clearly, the symmetry condition (P2) is satisfied. It is shown below in Claim 5 that if \(x_1 + x_2 - 1 > 0\) then either \(p_1 = A - tx^2\) or, \(p_1 = A - t(z - x_1)^2\). Claim 6 below shows that \(z > 1/2\) if \(x_1 + x_2 - 1 > 0\).

Thus, (P1)-(P4) hold under the Kalai–Smorodinski solution. So, by Proposition 1, both
firms locating at the center is an equilibrium. The other candidates for equilibrium are the symmetric ones \((x_1, 1-x_1)\) with \(x_1 \geq 1/4\). The next task is to examine which (if any) of these can be eliminated as an equilibrium.

Towards this, some further details are provided in the next section and the proof of Theorem 2 is given in section 9.

### 6.2 Some Claims and Their Proofs

The main purpose is to prove Claims 5 and 6. Some useful results are given first.

**Claim 1** Let \(x_1 < x_2\). In the derivation of \(M_1, \ v_2 > v_1\). In the derivation of \(M_2, \ w_1 > w_2\).

**Proof** To determine \(M_1\), one needs to maximize \(v_1\alpha\) subject to the constraint \(v_2(1-\alpha) = \Gamma_{2N}\). Since \(\alpha = [v_2 - v_1 + t(x_2^2 - x_1^2)]/[2t(x_2 - x_1)], \ v_2(1-\alpha) = \Gamma_{2N}\) can be written as

\[
v_2 \left[ 1 - \frac{x_1 + x_2}{2} \right] - \Gamma_{2N} = \frac{v_2(v_2 - v_1)}{2t(x_2 - x_1)}
\]

\(v_2 \geq 2t\). So, the LHS is greater than \(t(2 - x_1 - x_2) - t(x_2 - x_1) \geq 0\).

To determine \(M_2\), one has to maximize \(w_2(1-\beta)\) subject to the constraint \(w_1\beta = \Gamma_{1N}\). Since \(\beta = [w_2 - w_1 + t(x_2^2 - x_1^2)]/[2t(x_2 - x_1)], \ w_1\beta = \Gamma_{1N}\) can be written equivalently as

\[
w_1 \left[ \frac{x_1 + x_2}{2} \right] - \Gamma_{1N} = \frac{w_1(w_1 - w_2)}{2t(x_2 - x_1)}
\]

\(w_1 \geq 2t\). So, the LHS is greater than \(t(x_1 + x_2) - t(x_2 - x_1) \geq 0\).

**Claim 2** Let \(x_1 < x_2\) and \(x_1 + x_2 - 1 > 0\). Then, (i) \(w_2 = A - t(1-x_2)^2\), (ii) \(w_1 = A - (\beta-x_1)^2\)

\(\Rightarrow v_2 < w_1, \ (iii) v_2 \geq w_1 \Rightarrow v_2 - w_1 \leq t(1+x_1-x_2)(x_1+x_2-1), \ (iv) \alpha+\beta-1 < (x_1+x_2-1)/2\).

**Proof** (i) If \(w_2 = A - t(1-x_2)^2\) in the derivation of \(M_2\) then by the preceding claim, \(w_1 > w_2 = A - t(1-x_2)^2 > A - tx_1^2\) since \(x_1 + x_2 - 1 > 0\). This is a contradiction.

(ii) \(w_1\beta = \Gamma_{1N} \leq t/2\) and \(v_2(1-\alpha) = \Gamma_{2N} \leq t/2\). Both \(w_1\) and \(v_2\) are greater than or equal to 2t. So, \(\beta \leq 1/4\) and \(1-\alpha \leq 1/4\).

\[
\alpha + \beta - 1 = -((\Gamma_{2N}/v_2) + (\Gamma_{1N}/w_1)) \quad \text{and} \quad \Gamma_{2N} = (2t/3)(x_2-x_1)(x_1+x_2-1). \quad \text{So,}
\]

\[
\alpha + \beta - 1 = \frac{2t(x_2-x_1)}{3v_2}(x_1+x_2-1) + \frac{\Gamma_{1N}}{w_1v_2}(v_2-w_1) \quad (5)
\]

Suppose that \(w_1 = A - (\beta-x_1)^2\). Since \(v_2 \leq A - t(\alpha-x_2)^2, \ v_2 - w_1 \leq t(\alpha + \beta - x_1 - x_2)(x_2-x_1-\alpha + \beta)\). If \(\alpha + \beta - x_1 - x_2 \geq 0\) then \(v_2 - w_1 \leq t(x_2-x_1)(\alpha + \beta - x_1 - x_2) \leq \)
\( t(x_2 - x_1)(\alpha + \beta - 1) \). Substitution in (5) leads to \( 1 \leq [2t(x_2 - x_1)/(3v_2)] + [t(x_2 - x_1)\beta/v_2] \), a contradiction. If \( \alpha + \beta - x_1 - x_2 < 0 \) then \( \alpha < x_2 \) because \( \beta > x_1 \) and \( x_2 - x_1 - \alpha + \beta > 0 \). Therefore, \( v_2 < w_1 \).

(iii) It follows from (i) and (ii) that if \( v_2 \geq w_1 \) then \( w_1 = A - tx_1^2 \). Since \( v_2 \leq A - t(1-x_2)^2 \), \( v_2 - w_1 \leq t(1+x_1-x_2)(x_1+x_2-1) \).

(iv) If \( v_2 < w_1 \) then from (5), \( \alpha + \beta - 1 < (x_1 + x_2 - 1)/3 < (x_1 + x_2 - 1)/2 \).

If \( v_2 \geq w_1 \) then \( \alpha + \beta - 1 > 0 \). Since \( 2t(x_2 - x_1)/(3v_2) \leq 1/3 \) and \( t(1+x_1-x_2)\beta/v_2 \leq 1/8 \), it follows from (5) that \( \alpha + \beta - 1 < (x_1 + x_2 - 1)/2 \).

**Claim 3** Suppose that \( x_1 < x_2 \) and \( x_1 + x_2 - 1 > 0 \). Then \((M_1 - \Gamma_1N) - (M_2 - \Gamma_2N) > 0\).

**Proof** \( M_1 = v_1\alpha \) and \( M_2 = w_2(1-\beta) \). Therefore, \((M_1 - \Gamma_1N) - (M_2 - \Gamma_2N) = v_1\alpha + v_2(1-\alpha) - w_1\beta - w_2(1-\beta) \). Since \( v_2 = v_1 + tx_2 - x_1(2\alpha - x_2 - x_1) \) and \( w_2 = w_1 + tx_2 - x_1(2\beta - x_1 - x_2) \), \((M_1 - \Gamma_1N) - (M_2 - \Gamma_2N)\) equals

\[
(v_1 - v_2)\alpha + v_2 - w_1 - t(x_2 - x_1)(2\beta - x_1 - x_2)(1 - \beta)
\]

\[
= v_2 - w_1 + t(x_2 - x_1)(x_1 + x_2 - 2\alpha)\alpha + t(x_2 - x_1)(x_1 + x_2 - 2\beta)(1 - \beta)
\]

\[
= v_2 - w_1 + t(x_2 - x_1)[(x_1 + x_2 + 1 - 2\alpha - 2\beta)(\alpha - \beta) + x_1 + x_2 - \alpha - \beta]
\]

\[
= v_2 - w_1 + t(x_2 - x_1)[(x_1 + x_2 + 1 - 2\alpha - 2\beta)((1/2) + \alpha - \beta) + (1/2)(x_1 + x_2 - 1)]
\]

From Claim 2, this is positive if \( v_2 \geq w_1 \).

If \( v_2 < w_1 \) then \( v_1 < v_2 < w_1 \leq A - tx_1^2 < A - t(1-x_2)^2 \). So, \( v_2 = A - t(\alpha - x_2)^2 \). Moreover, \( w_1 \leq A - t(\beta - x_1)^2 \), so \( v_2 - w_1 \geq t(\beta - x_1)^2 - t(\alpha - x_2)^2 = t(x_1 + x_2 - \alpha - \beta)(x_2 - x_1 - \alpha + \beta) \). Therefore, \((M_1 - \Gamma_1N) - (M_2 - \Gamma_2N)\) is greater than or equal to

\[
t(x_2 - x_1)(x_1 + x_2 + 1 - 2\alpha - 2\beta)(\alpha - \beta) + t(x_1 + x_2 - \alpha - \beta)(\alpha - \beta)
\]

This is positive since \( \alpha + \beta - 1 < (x_1 + x_2 - 1)/2 \).

The expression for \((M_1 - \Gamma_1N) - (M_2 - \Gamma_2N)\) when \( w_1 = A - tx_1^2 \) and \( v_2 = A - t(1-x_2)^2 \) is developed below for later usage.

In this case \( v_2 - w_1 = t(1+x_1-x_2)(x_1+x_2-1) \). From (5),

\[
x_1 + x_2 + 1 - 2\alpha - 2\beta = x_1 + x_2 - 1 - \frac{4t(x_2 - x_1)(x_1 + x_2 - 1)}{3v_2} - \frac{2\beta}{v_2}(v_2 - w_1)
\]

\[
= (x_1 + x_2 - 1) \left[ 1 - \frac{4t(x_2 - x_1)}{3v_2} - \frac{2(1+x_1-x_2)\beta}{v_2} \right]
\]
Let
\[
f = \left[ 1 - \frac{4t(x_2 - x_1)}{3v_2} - \frac{2t(1 + x_1 - x_2)\beta}{v_2} \right] \left( \frac{1}{2} + \alpha - \beta \right) - \frac{1}{2}
\]  

(6)

Then
\[
M_1 - \Gamma_1N - M_2 + \Gamma_2N
= t(1 + x_1 - x_2)(x_1 + x_2 - 1) + t(x_2 - x_1)(x_1 + x_2 - 1)(1 + f)
= t(x_1 + x_2 - 1)[1 + (x_2 - x_1)f]
\]

Let \( K = \frac{1 + (x_2 - x_1)f}{M_2 - \Gamma_2N} \). Since \( M_2 - \Gamma_2N = w_2(1 - \beta) - \Gamma_2N = [w_1 - t(x_2 - x_1)(x_1 + x_2 - 2\beta)](1 - \beta) - \Gamma_2N = A - tx_1^2 - \Gamma_1N - \Gamma_2N - t(x_2 - x_1)(x_1 + x_2 - 2\beta)(1 - \beta) \),
\[
K = \frac{1 + (x_2 - x_1)f}{A - tx_1^2 - \Gamma_1N - \Gamma_2N - t(x_2 - x_1)(x_1 + x_2 - 2\beta)(1 - \beta)}
\]  

(7)

and \( M = 1 + t(x_1 + x_2 - 1)K \).

**Claim 4** Suppose that \( x_1 < x_2 \) and \( x_1 + x_2 - 1 > 0 \). Then \( M < (x_1 + x_2)/(2 - x_1 - x_2) \).

**Proof** If \( v_2 < w_1 \) then using (5) and collecting the terms together involving \( v_2 - w_1 \) it can be shown that \( (M_1 - \Gamma_1N) - (M_2 - \Gamma_2N) \leq t(x_2 - x_1)(x_1 + x_2 - 1)[(1/2) + \alpha - \beta] + (t/2)(x_2 - x_1)(x_1 + x_2 - 1) \leq 2t(x_2 - x_1)(x_1 + x_2 - 1) \).

If \( v_2 \geq w_1 \) then \( \alpha + \beta - 1 > 0 \) and \( v_2 - w_1 \leq t(1 + x_1 - x_2)(x_1 + x_2 - 1) \). So, \( (M_1 - \Gamma_1N) - (M_2 - \Gamma_2N) \leq t(1 + x_1 - x_2)(x_1 + x_2 - 1) + t(x_2 - x_1)(x_1 + x_2 - 1)[(1/2) + \alpha - \beta] + (t/2)(x_2 - x_1)(x_1 + x_2 - 1) \leq 2t(x_1 + x_2 - 1) \).

One needs to show that \( (M_1 - \Gamma_1N)(2 - x_1 - x_2) < (M_2 - \Gamma_2N)(x_1 + x_2) \), i.e.,
\[
(x_1 + x_2 - 1)(M_1 - \Gamma_1N + M_2 - \Gamma_2N) - [(M_1 - \Gamma_1N) - (M_2 - \Gamma_2N)] > 0
\]

Since \( (M_1 - \Gamma_1N) - (M_2 - \Gamma_2N) \leq 2t(x_1 + x_2 - 1) \), it needs to be shown that \( M_1 - \Gamma_1N + M_2 - \Gamma_2N > 2t \). It is enough to show that \( M_2 - \Gamma_2N \geq t \). All the prices in the determination of \( M_2 \) are greater than or equal to \( 2t \), so the total profit \( M_2 + \Gamma_1N \geq 2t \). Both \( \Gamma_1N \) and \( \Gamma_2N \) are less than or equal to \( t/2 \), so \( M_2 - \Gamma_2N \geq t \). Therefore, \( M < (x_1 + x_2)/(2 - x_1 - x_2) \).

The following two claims deal with prices which occur in the determination of actual profits \( \Gamma_1 \) and \( \Gamma_2 \).
Claim 5 Suppose that \(x_1 < x_2\) and \(x_1 + x_2 - 1 > 0\). Then \(p_2 = A - t(1 - x_2)^2\) cannot hold. Therefore, either \(p_1 = A - tx_1^2\) or \(p_1 = A - t(z - x_1)^2\).

**Proof** By the preceding Claim, \(M < (x_1 + x_2)/(2 - x_1 - x_2)\). Since \(x_1 + x_2 - 1 > 0\), \(\Gamma_1 N/\Gamma_2 N < (x_1 + x_2)/(2 - x_1 - x_2)\) as well. Therefore, from (4), \(\Gamma_1/\Gamma_2 < (x_1 + x_2)/(2 - x_1 - x_2)\).

If \(p_2 = A - t(1 - x_2)^2 > A - tx_1^2 \geq p_1\) then \(\Gamma_1/\Gamma_2 \geq (x_1 + x_2)/(2 - x_1 - x_2)\), a contradiction. Therefore, if \(x_1 + x_2 - 1 > 0\) then either \(p_1 = A - tx_1^2\) or \(p_1 = A - t(z - x_1)^2\).

Claim 6 Suppose that \(x_1 < x_2\) and \(x_1 + x_2 - 1 > 0\). Then \(z > 1/2\).

**Proof** \(\Gamma_1 = p_1 z\) and \(\Gamma_2 = p_2(1 - z) = [p_1 + t(x_2 - x_1)(2z - x_1 - x_2)](1 - z)\). By the preceding Claim, \(p_1 = A - tx_1^2\) or \(p_1 = A - t(z - x_1)^2\). In either case, \(\Gamma_1\) is increasing and \(\Gamma_2\) is decreasing in \(z\). Therefore, \((\Gamma_1 - \Gamma_1 N)/(\Gamma_2 - \Gamma_2 N)\) is an increasing function of \(z\). When \(z = 1/2\), \((\Gamma_1 - \Gamma_1 N)/(\Gamma_2 - \Gamma_2 N) < 1\). Hence, \((\Gamma_1 - \Gamma_1 N)/(\Gamma_2 - \Gamma_2 N) < 1\) if \(z \leq 1/2\). By Claim 3, \(1 < M\). So, \((\Gamma_1 - \Gamma_1 N)/(\Gamma_2 - \Gamma_2 N) < M\), a contradiction. So, \(z > 1/2\) if \(x_1 + x_2 - 1 > 0\).
7 Behavior of the Ratio of Profits under the Kalai–Smorodinski Bargaining Solution

The objective at present is to determine which of the symmetric locations in [1/4, 3/4] is an equilibrium. This requires a closer look at the ratio of profits as captured by $M$. To begin with, the prices which determine $M_1$ and $M_2$ and the derivative of $M$ are examined. Subsequently, the properties of $M$ are determined.

7.1 Properties of $M$

The prices which determine $M_1$ and $M_2$ can be of three types, depending on the location of the reservation price consumer. Suppose that $x_1 < x_2 \leq 3/4$ and $x_1 + x_2 - 1 \geq 0$. Since both firms are inside the market quartiles, the reservation price consumer is either at zero or at one.

Consider $M_2$ first. This corresponds to a profit of $\Gamma_{1N}$ for Firm 1. Suppose one pays the reservation price. Then $w_2 = A - t(1 - x_2)^2 \geq A - tx_2^2 \geq w_1$. Therefore, the market share of Firm 1 is at least $(x_1 + x_2)/2 \geq 1/2$. Since $w_1 \geq A - t \geq 2t$, the profit of Firm 1 is at least $t$. Clearly, this is greater than $\Gamma_{1N}$, a contradiction. So, one does not pay the reservation price. Therefore, zero must pay the reservation price, i.e., if $x_1 < x_2 \leq 3/4$ and $x_1 + x_2 - 1 \geq 0$ then $w_1 = A - tx_2^2$ in the determination of $M_2$.

In contrast, two of the prices are involved in the determination of $M_1$ in the sense that at some locations zero pays the reservation price and at some other locations one pays the reservation price. These locations can be characterized as follows.

Suppose that $x_1 < x_2 \leq 3/4$ and $x_1 + x_2 - 1 \geq 0$. Let $\bar{x}_1 \in (1 - x_2, 1/2)$ satisfy

$$9[A - t(1 - x_2)^2](1 - 2\bar{x}_1) - t(x_2 - \bar{x}_1)^2(4 - \bar{x}_1 - x_2)^2 = 0 \quad (8)$$

If $x_1 \leq \bar{x}_1$ then $v_2 = A - t(1 - x_2)^2$ and if $x_1 > \bar{x}_1$ then $v_1 = A - tx_2^2$ in the derivation of $M_1$.

In particular, at symmetric and near symmetric locations $v_2 = A - t(1 - x_2)^2$. The details of derivation of (8) are given in the last subsection of this section. The intuition is as follows. $M_1$ corresponds to the maximum profit of Firm 1 when the profit of Firm 2 is $\Gamma_{2N}$. Since the profit of Firm 2 is fixed, a higher price for Firm 2 means a lower market share for Firm 2. This in turn gives a higher market share and profit to Firm 1. Therefore, in the derivation of $M_1$, the price of Firm 2 should be as high as possible under the circumstances.

The price of Firm 2 is at most $A - t(1 - x_2)^2$. Therefore, at symmetric and near symmetric
locations $v_2 = A - t(1 - x_2)^2$. However, if Firm 1 moves towards its rival and $x_1 \geq 1/2$ then zero must pay the reservation price since the entire market is served, i.e., $v_1 = A - tx_1^2$. (8) says that the switch between the reservation price consumer from one to zero occurs somewhere to the left of $1/2$, i.e., when $x_1 = \bar{x}_1$.

The next two claims deal with the derivatives of $M$ for selected locations. These are proved in the last subsection of this section.

**Claim 7** Let $x_1 < x_2 \leq 2/3$ and $x_1 + x_2 - 1 \geq 0$. Suppose that $w_1 = v_1 = A - tx_1^2$. Then $M$ is a decreasing function of $x_1$.

It was shown in the previous section that when $w_1 = A - tx_1^2$ and $v_2 = A - t(1 - x_2)^2$, $M = 1 + t(x_1 + x_2 - 1)K$ where $K$ is defined in (7).

**Claim 8** Let $x_1 < x_2 \leq 2/3$ and $x_1 + x_2 - 1 \geq 0$. Then $K$ is a decreasing function of $x_1$.

### 7.2 Properties of $\hat{M}$

**Claim 9** Let $x_1 < x_2 \leq 3/4$ and $x_1 + x_2 - 1 \geq 0$. Then $\hat{K}$ is an increasing function of $x_1$.

This is proved in the next subsection. The numerator of $\hat{K}$ is increasing and the denominator is decreasing in $x_1$. It follows that $\hat{M}$ is an increasing function of $x_1$ in the relevant region.

### 7.3 Proofs of the Claims

Equation (8) is obtained as follows.

Let $x_1 < x_2 \leq 3/4$ and $x_1 + x_2 - 1 \geq 0$. The LHS of (8) is decreasing in $\bar{x}_1$, is negative when $\bar{x}_1 = 1/2$ and positive when $\bar{x}_1 = 1 - x_2$. So, $\bar{x}_1$ is unique. The prices $v_1$ and $v_2$ satisfy $v_1 = v_2 = 2t(x_2 - x_1)[1 - (\Gamma_{2N}/v_2)] + t(x_2^2 - x_1^2)$ since the market share of Firm 2 is $\Gamma_{2N}/v_2$. Setting $v_1 = A - tx_1^2$ and $v_2 = A - (1 - x_2)^2$ and solving for $x_1$ yields (8), i.e., $x_1 = \bar{x}_1$. For fixed $v_2$, $v_2 = 2t(x_2 - x_1)[1 - (\Gamma_{2N}/v_2)] + t(x_2^2 - x_1^2)$ is increasing in $x_1$. If $x_1 > \bar{x}_1$ and $v_2 = A - t(1 - x_2)^2$ then $v_1$ must increase, i.e., $v_1 > A - tx_1^2$, a contradiction. If $x_1 < \bar{x}_1$ and $v_2 < A - t(1 - x_2)^2$ then $v_1$ must decrease, i.e., $v_1 < A - tx_1^2$ and no consumer pays the reservation price, a contradiction. So, If $x_1 \leq \bar{x}_1$ then $v_2 = A - t(1 - x_2)^2$ and if $x_1 > \bar{x}_1$ then $v_1 = A - tx_1^2$ in the derivation of $M_1$. 

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The derivatives with respect to $x_1$ are denoted by the prime symbol.

**Proof of Claim 7** Since $\Gamma_{1N} = (t/18)(x_2 - x_1)(2 + x_1 + x_2)^2$, $\Gamma'_{1N} = -(t/18)(2 + x_1 + x_2)(2 + 3x_1 - x_2)$. Similarly, $\Gamma_{2N} = (t/18)(x_2 - x_1)(4 - x_1 - x_2)^2$ and $\Gamma'_{2N} = -(t/18)(4 - x_1 - x_2)(4 - 3x_1 + x_2)$. It follows that $|\Gamma'_{2N}| > \Gamma_{2N}$ and since $x_2 \leq 2/3$, $|\Gamma'_{1N}| > \Gamma_{1N}$.

In this case, $w_1 = v_1 = A - tx_2^2$, $w_2 = A - tx_2^2 + 2t(x_2 - x_1)\beta$ and $v_2 = A - tx_2^2 + 2t(x_2 - x_1)\alpha$. $M_2 - \Gamma_{2N} = w_2(1 - \beta) - v_2(1 - \alpha) = (A - tx_2^2)(\alpha - \beta) + 2t(x_2 - x_1)(\alpha + \beta - 1)(\alpha - \beta)$. $M_1 - \Gamma_{1N} = v_1\alpha - w_1\beta = (A - tx_2^2)(\alpha - \beta)$. Therefore,

$$M = \frac{A - tx_2^2}{A - tx_2^2 + 2t(x_2 - x_1)(\alpha + \beta - 1)}$$

$$[A - tx_2^2 + 2t(x_2 - x_1)(1 - \alpha)](1 - \alpha) = \Gamma_{2N} \text{ or, } -[A - tx_2^2 - 2t(x_2 - x_1)(1 - 2\alpha)]\alpha' - 2t\alpha(1 - \alpha) = \Gamma'_{2N} \text{, } v_2 \geq 2t \text{ means } 2t(1 - \alpha) \leq \Gamma_{2N} \text{ which shows that } \alpha' \text{ is positive. Since } \beta = \Gamma_{1N} / (A - tx_2^2),$$

$$\beta' = \frac{(A - tx_2^2)\Gamma'_{1N} + 2tx_1\Gamma_{1N}}{(A - tx_2^2)^2}$$

This is negative because $|\Gamma'_{1N}| > \Gamma_{1N}$. Moreover, $0 \leq -(x_2 - x_1)\beta' \leq -(x_2 - x_1)\Gamma'_{1N} / (A - tx_2^2) \leq \Gamma_{1N} / (A - tx_2^2) \leq \beta$.

The sign of $M'$ is determined by

$$-x_1[A - tx_2^2 + 2t(x_2 - x_1)(\alpha + \beta - 1)] + (A - tx_1^2)[\alpha + \beta - 1 - (x_2 - x_1)(\alpha' + \beta')]$$

Since $\alpha + \beta - 1$ is positive and $\alpha - 1 - (x_2 - x_1)\alpha'$ is negative it suffices to show that $-x_1(A - tx_2^2) + (A - tx_1^2)[\beta - (x_2 - x_1)\beta'] < 0$.

$$(A - tx_1^2)[\beta - (x_2 - x_1)\beta'] \leq 2(A - tx_1^2)\beta = 2\Gamma_{1N} \leq t/2. \text{ On the other hand, } (A - tx_2^2)x_1 > t/2. \text{ So, the inequality holds.}$$

**Proof of Claim 8** In this case, $w_1 = A - tx_1^2$, $v_2 = A - t(1 - x_2)^2$ and $w_2 = A - tx_2^2 + 2t(x_2 - x_1)\beta$.

Since $(A - tx_1^2)\beta = \Gamma_{1N}$, the denominator of $K$ is $|A - tx_2^2 + 2t(x_2 - x_1)\beta|(1 - \beta) - \Gamma_{2N}$. Its derivative is $-2t\beta + 2t(x_2 - x_1)\beta' - \Gamma'_{2N} = -[w_2 - 2t(x_2 - x_1)(1 - \beta)]\beta' - 2t\beta(1 - \beta) - \Gamma'_{2N}$. It is shown below that $\beta \leq 1/12$. $-\Gamma'_{2N} = (t/18)(4 - x_1 - x_2)(4 - 3x_1 + x_2) > t/6$.

Since $\beta'$ is negative the denominator is increasing.

The numerator of $K$ is $1 + (x_2 - x_1)f$. Its derivative is $-f + (x_2 - x_1)f'$. It can be shown that $f + (x_2 - x_1)f' \leq 1$ and $f > 1/2$. Therefore, $-f + (x_2 - x_1)f' \leq 1 - 2f < 0$.

$$f + (x_2 - x_1)f'$$

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Now, 0 ≤ (x₂ - x₁)β ≤ β. 1 - α = Γ₂N/v₂. So, α' = -Γ₂N/v₂. One can show that
(x₂ - x₁)(α' - β') - (1 - α + β) < 0. So, f + (x₂ - x₁)f' ≤ 1.

To show f > 1/2, both Γ₁N and Γ₂N are less than or equal to t/6 because x₂ - x₁ ≤ 1/3. So, Γ₁N/w₁ and Γ₂N/v₂ are less than or equal to 1/12. (1/2) + α - β = (3/2) - (Γ₂N/v₂) - (Γ₁N/w₁) ≥ (3/2) - (1/6) = 4/3. Since 1/2 ≤ x₂ and A ≥ 3t, v₂ = A - t(1 - x₂)² ≥ 11t/4. Therefore, 4t(x₂ - x₁)/(3v₂) ≤ 4t/(99t/4) = 16/99. 2t/v₂ ≤ 8/11 and 2t/v₂ ≤ (8/11)(1/12)

**Proof of Claim 9**  The numerator of \( \tilde{K} \) is 2 - (x₂ - x₁)[(7/3) + (1 + x₁ - x₂)g]. Its derivative is
(7/3) + (1 + x₁ - x₂)g' - (x₂ - x₁)[g + (1 + x₁ - x₂)g'] = (7/3) + (1 + 2x₁ - 2x₂)g - (x₂ - x₁)(1 + x₁ - x₂)g'.

\[
g' = \frac{t(A - tx_1^2)(1 - 2\sigma') + 2tx_1g(A - tx_1^2)}{(A - tx_1^2)^2} = \frac{t(1 - 2\sigma') + 2tx_1g}{A - tx_1^2}
\]

0 < σ' = tx₁[A - t(1 - x₂)²]/[(A - tx₁²)²] = 2tx₁σ/(A - tx₁²) < 1. Since σ ≥ 1/2,
x₁ + x₂ + 1 - 2σ ≤ 3/2 and hence, g ≤ 3/4. Therefore, g' ≤ (5/2)t/(A - tx₁²) ≤ 5/4. Since
(x₂ - x₁)(1 + x₁ - x₂) ≤ 1/4, the derivative is greater than 2.

Ignoring 2, the denominator of \( \tilde{K} \) is [A - tx₁² + 2t(x₂ - x₁)σ](1 - σ) - Γ₂N. Its derivative
with respect to x₁ is

\[-[A - tx₁² + 2t(x₂ - x₁)(2σ - 1)]σ' - 2tσ(1 - σ) - Γ₂N'
\]

It is enough to show that -\((A - tx₁²)σ' - 2tσ(1 - σ) - Γ₂N' < 0. -\((A - tx₁²)σ'' - 2tσ'(1 - 2σ) - Γ₂N' < 0 because σ'' > 0, 2σ - 1 = t(x₁ + x₂)²/(A - tx₁²), x₁ + x₂ - 1 ≤ 1/2 and Γ₂N' = (t/9)(8 - 3x₁ - x₂) ≥ 5t/9. Therefore, it suffices to show that -\((A - tx₁²)σ' - 2tσ(1 - σ) - Γ₂N' < 0 at symmetric locations, i.e., -\([A - tx₁²]t(1 - x₂)/[A - t(1 - x₂)²] - (t/2) + (t/6)(1 + 4x₂) < 0. Since x₂ ≤ 3/4 and \((A - tx₁²)/[A - t(1 - x₂)²] > 2/3 the inequality holds.
8 An Inequality at Symmetric Locations

The difference \( K - \tilde{K} \) at symmetric locations plays a key role in the proof of Theorem 2. This can be evaluated at symmetric locations from the respective expressions. Informally, the sign of \( K - \tilde{K} \) is the same as the LHS of (2) in Theorem 2. The next Claim provides a precise statement.

**Claim 10** Let \((x_1, x_2)\) be a pair of symmetric locations with \(1/4 \leq x_1 < 1/2\). Then \(6(A - tx_1^2)^3(M_2 - \Gamma_2(N))(K - \tilde{K})/t^3\) equals

\[
\left(\frac{A}{t} - x_1^2\right)^3(14 - 40x_1) - \left(\frac{A}{t} - x_1^2\right)^2(1 - 2x_1)(20 - 76x_1) + \left(\frac{A}{t} - x_1^2\right)(1 - 2x_1)^2(9 - 60x_1) + 18x_1(1 - 2x_1)^3
\]

So, the sign of this expression determines the sign of \( K - \tilde{K} \). Importantly, this is the LHS of (2) in Theorem 2.

Throughout this section symmetry is assumed.

\[
K = \frac{1 + (x_2 - x_1)f}{A - tx_1^2 - \Gamma_1N - \Gamma_2N - t(x_2 - x_1)(x_1 + x_2 - 2\beta)(1 - \beta)}
\]

\[
\tilde{K} = \frac{2 - (x_2 - x_1)[(7/3) + (1 + x_1 - x_2)g]}{2[A - tx_1^2 + t(x_2 - x_1)(2\sigma - x_1 - x_2)](1 - \sigma) - 2\Gamma_2N}
\]

Since \(\Gamma_2N = (t/18)(x_2 - x_1)(4 - x_1 - x_2)^2 = (t/2)(1 - 2x_1),\) \(2\Gamma_2N = t(1 - 2x_1).\) Since \(\sigma\) equals 1/2, the denominator of \(\tilde{K}\) is \(A - tx_1^2 - 2\Gamma_1N = A - tx_1^2 - t(1 - 2x_1) = A - t(1 - x_1)^2.\)

The denominator of \(K\) is \(A - tx_1^2 - 2\Gamma_1N - t(1 - 2x_1)(1 - 2\beta)(1 - \beta),\) \(1 - 2\beta = (A - tx_1^2 - 2\Gamma_1N)/(A - tx_1^2).\) So, the denominator of \(K\) is

\[
[A - t(1 - x_1)^2]\left[1 - \frac{t(1 - 2x_1)(1 - \beta)}{A - tx_1^2}\right]
\]

Therefore, \((M_2 - \Gamma_2(N))(K - \tilde{K})\) is

\[
1 + (1 - 2x_1)f - \left[1 - \frac{t(1 - 2x_1)(1 - \beta)}{A - tx_1^2}\right]\left[2 - \frac{7(1 - 2x_1)}{3} - 2x_1(1 - 2x_1)g\right] = \frac{1}{2}(1 + 2x_1) + (1 - 2x_1)\left(f + \frac{1}{2}\right) + \left[1 - \frac{t(1 - 2x_1)(1 - \beta)}{A - tx_1^2}\right]\left[1 - \frac{14x_1}{3} + \frac{2tx_1(1 - 2x_1)}{A - tx_1^2}\right]
\]
Multiplication with $6(A - tx_i^3)$ gives
\[
3(A - tx_i^3)^3(1 + 2x_1) + 6(A - tx_i^3)^3(1 - 2x_1) \left(f + \frac{1}{2}\right)
+ 2(A - tx_i^3) \left[A - tx_i^2 - t(1 - 2x_1)(1 - \beta)\right] \left[(A - tx_i^2)(1 - 14x_1) + 6tx_1(1 - 2x_1)\right]
\]

Since $\beta = \Gamma_{1N}/(A - tx_i^3) = t(1 - 2x_1)/[2(A - tx_i^3)]$, this is the same as
\[
3(A - tx_i^3)^3(1 + 2x_1) + 6(A - tx_i^3)^3(1 - 2x_1) \left(f + \frac{1}{2}\right)
+ 2(A - tx_i^3) \left[A - tx_i^2 - t(1 - 2x_1)\right] \left[(A - tx_i^2)(1 - 14x_1) + 6tx_1(1 - 2x_1)\right]
+ t^2(1 - 2x_1)^2 \left[(A - tx_i^2)(1 - 14x_1) + 6tx_1(1 - 2x_1)\right]
\]

At symmetric locations $v_2 = A - tx_i^3$ and from (6),
\[
f + \frac{1}{2} = \left[1 - 4t(1 - 2x_1)\right] \left(\frac{3}{2} - \frac{2\Gamma_{1N}}{v_2}\right)
\]

Therefore, $6(A - tx_i^3)^3[f + (1/2)]$ equals
\[
\left[3(A - tx_i^3)^2 - 4t(1 - 2x_1)(A - tx_i^3) - 12tx_1\Gamma_{1N}\right] \left[3(A - tx_i^3) - 4\Gamma_{1N}\right]
= \left[3(A - tx_i^3)^2 - 4t(1 - 2x_1)(A - tx_i^3) - 6t^2x_1(1 - 2x_1)\right] \left[3(A - tx_i^3) - 2t(1 - 2x_1)\right]
\]

Therefore, $6(A - tx_i^3)^3(M_2 - \Gamma_{2N})(K - \tilde{K})$ equals
\[
3(A - tx_i^3)^3(1 + 2x_1) + 3(1 - 2x_1)(A - tx_i^3)^2 \left[3(A - tx_i^3) - 2t(1 - 2x_1)\right]
+ (1 - 2x_1) \left[4t(1 - 2x_1)(A - tx_i^3) + 6t^2x_1(1 - 2x_1)\right] \left[3(A - tx_i^3) - 2t(1 - 2x_1)\right]
+ 2(A - tx_i^3) \left[A - tx_i^2 - t(1 - 2x_1)\right] \left[(A - tx_i^2)(1 - 14x_1) + 6tx_1(1 - 2x_1)\right]
+ t^2(1 - 2x_1)^2 \left[(A - tx_i^2)(1 - 14x_1) + 6tx_1(1 - 2x_1)\right]
\]
Division by \( t^3 \) gives

\[
3 \left( \frac{A}{t} - x_1^2 \right)^3 (1 + 2x_1) + 3(1 - 2x_1) \left( \frac{A}{t} - x_1^2 \right)^2 \left[ 3 \left( \frac{A}{t} - x_1^2 \right) - 2(1 - 2x_1) \right] \\
- \ (1 - 2x_1) \left[ 4(1 - 2x_1) \left( \frac{A}{t} - x_1^2 \right) + 6x_1(1 - 2x_1) \right] \left[ 3 \left( \frac{A}{t} - x_1^2 \right) - 2(1 - 2x_1) \right] \\
+ \ 2 \left( \frac{A}{t} - x_1^2 \right) \left( \left( \frac{A}{t} - x_1^2 \right) - (1 - 2x_1) \right) \left[ (\frac{A}{t} - x_1^2) (1 - 14x_1) + 6x_1(1 - 2x_1) \right] \\
+ \ (1 - 2x_1)^2 \left[ \left( \frac{A}{t} - x_1^2 \right) (1 - 14x_1) + 6x_1(1 - 2x_1) \right]
\]

= \ 12(1 - x_1) \left( \frac{A}{t} - x_1^2 \right)^3 - 6(1 - 2x_1)^2 \left( \frac{A}{t} - x_1^2 \right)^2 \\
- \ 12(1 - 2x_1)^2 \left( \frac{A}{t} - x_1^2 \right)^2 + (1 - 2x_1)^2(8 - 34x_1) \left( \frac{A}{t} - x_1^2 \right) + 12x_1(1 - 2x_1)^3 \\
+ \ 2(1 - 14x_1) \left( \frac{A}{t} - x_1^2 \right)^3 + (1 - 2x_1)(-2 + 40x_1) \left( \frac{A}{t} - x_1^2 \right)^2 - 12x_1(1 - 2x_1)^2 \left( \frac{A}{t} - x_1^2 \right) \\
+ \ (1 - 2x_1)^2(1 - 14x_1) \left( \frac{A}{t} - x_1^2 \right) + 6x_1(1 - 2x_1)^3
\]

This is the same as

\[
\left( \frac{A}{t} - x_1^2 \right)^3 (14 - 40x_1) - \left( \frac{A}{t} - x_1^2 \right)^2 (1 - 2x_1)(20 - 76x_1) \\
+ \left( \frac{A}{t} - x_1^2 \right) (1 - 2x_1)^2 (9 - 60x_1) + 18x_1(1 - 2x_1)^3
\]

This is the LHS of (2) in Theorem 2.
9 Proof of Theorem 2

It was noted earlier that $(1/2, 1/2)$ is an equilibrium and the other candidates for equilibrium are the symmetric ones in the interval $[1/4, 3/4]$. Any such symmetric pair is an equilibrium if for fixed $x_2 \in (1/2, 3/4)$, $\Gamma_1(x_1, x_2) < \Gamma_1(1 - x_2, x_2)$ for $x_1 < 1 - x_2$ and for $x_1 > 1 - x_2$. By Lemma 1 it suffices to consider the case $x_1 > 1 - x_2$.

Let $x_1 < x_2$, $1/2 < x_2 \leq 3/4$ and $x_1 + x_2 - 1 \geq 0$. Then $p_1 = A - tx_2^2$ in the Kalai-Smorodinski solution (Claim 5). Therefore, $p_1\sigma = (A - tx_2^2)\sigma = [A - t(1 - x_2)^2]/2$. This is the profit of Firm 1 when the two firms are located symmetrically at $(1 - x_2, x_2)$. Thus, if the prices and market share $(p_1, p_2, z)$ at $(x_1, x_2)$ in the Kalai-Smorodinski solution satisfies $z < \sigma$, then $\Gamma_1 = p_1z < p_1\sigma = [A - t(1 - x_2)^2]/2$ and vice versa.

Since $p_1 = A - tx_2^2$ and $p_2 = p_1 + t(x_2 - x_1)(2z - x_1 - x_2)$,

$$M = \frac{\Gamma_1 - \Gamma_{1N}}{\Gamma_2 - \Gamma_{2N}} = \frac{(A - tx_2^2)z - \Gamma_{1N}}{[A - tx_2^2 + t(x_2 - x_1)(2z - x_1 - x_2)](1 - z) - \Gamma_{2N}}$$

The RHS of the above equation is a strictly increasing function of $z$. Hence, $z < \sigma$ iff

$$M < \frac{(A - tx_2^2)\sigma - \Gamma_{1N}}{[A - tx_2^2 + t(x_2 - x_1)(2\sigma - x_1 - x_2)](1 - \sigma) - \Gamma_{2N}} = \tilde{M}$$

This is the important element of the proof.

Consider a pair of symmetric locations $(\hat{x}_1, \hat{x}_2)$ with $\hat{x}_1 \geq 1/4$. Suppose that $\hat{x}_1 < x_2^1$ in (2). Then $K > \hat{K}$ at the pair of symmetric locations (from Claim 10). Hence, at near symmetric locations $(\hat{x}_1 + \epsilon, \hat{x}_2)$, $K > \hat{K}$ as well. This implies that $M > \tilde{M}$. So such symmetric pairs cannot be equilibria.

Now suppose that $\hat{x}_1 \geq x_1^1$, i.e., \(\hat{x}_2 = 1 - \hat{x}_1 \leq 1 - x_1^1 \leq 2/3\). Then $K \leq \hat{K}$ at $(\hat{x}_1, \hat{x}_2)$. Let $x_1 > \hat{x}_1$. First suppose that $x_1 \leq \bar{x}_1$ where $\bar{x}_1$ is as defined in (8), given $\hat{x}_2$. Then from Claims 8 and 9, $K$ is decreasing and $\hat{K}$ is increasing in $x_1$. Therefore, $M < \tilde{M}$. In particular, at the locations $(\bar{x}_1, \hat{x}_2)$, $M < \tilde{M}$. If $x_1 > \bar{x}_1$ then by Claim 7, $M$ is decreasing in $x_1$. Since $\tilde{M}$ is increasing in $x_1$, $M < \tilde{M}$.

This shows that $M < \tilde{M}$ for all $x_1 > \hat{x}_1$. Therefore, the pair of symmetric locations $(\hat{x}_1, \hat{x}_2)$ is an equilibrium. This proves the Theorem.

Consider an equilibrium $(1 - x_2, x_2)$ off the center, $x_2 \neq 1/2$. If Firm 1 moves to the right from this symmetric pair of locations its profit decreases. Then the question emerges if Firm 1 does not gain by moving to the right why does it not gain by moving to the left? As
was shown in section 3, a move to the left decreases the profit as well. This is why the pair of symmetric locations off the center emerges as equilibrium.

The derivative of the profit function as the firm approaches a symmetric location is positive from the left and can be negative from the right. So, the profit function can be nondifferentiable at a pair of symmetric locations. The change of sign of the derivative on opposite sides of the symmetric location is critical in ensuring an equilibrium off the center.
10 On the Convexity of the Set of Equilibria

10.1 An Example

Suppose that a solution satisfies (P1)–(P4) and a pair of symmetric locations \((x_1^*, x_2^*)\), \(1/4 \leq x_1^* < 1/2\) is an equilibrium. Then is it true that another pair of symmetric locations \((y_1^*, y_2^*)\), \(x_1^* < y_1^* < 1/2\) is also an equilibrium?

As the following example suggests the answer to this question is in the negative. (P1)–(P4) are not sufficient to ensure that the set of equilibria is convex. Additional conditions are required to ensure this.

Suppose that the firms charge the prices determined by the Kalai–Smorodinski bargaining solution if both the firms are located in the open interval \((0.3, 0.7)\) and charge the prices determined by the egalitarian solution otherwise.

Both the Kalai–Smorodinski solution and the egalitarian solution satisfy (P1)–(P4). So, this solution also satisfies (P1)–(P4). The set of equilibria under this solution is not convex.

Any pair of symmetric locations \((x_1^*, x_2^*)\), \(1/4 \leq x_1^* \leq 3/10\) is an equilibrium (since the prices are determined by the egalitarian solution). The set of equilibria under the Kalai–Smorodinski bargaining solution depends on \(A/t\), the ratio of the reservation price and the transportation cost parameter. Any such symmetric pair also emerges as an equilibrium under the new solution. There is no other equilibrium.

The set of symmetric equilibria under the Kalai–Smorodinski bargaining solution is contained in \([0.35, 0.65]\). In particular, the symmetric pair \([0.32, 0.68]\) is not an equilibrium under the new solution, i.e., the set of equilibria is not convex.

10.2 A Sufficient Condition

The question is, if a pair of symmetric locations \((x_1^*, x_2^*)\) is an equilibrium, Then when is it that another pair of symmetric locations \((y_1^*, y_2^*)\), \(x_1^* < y_1^*\) is also an equilibrium? Claim 12 below provides a sufficient condition.

**Claim 11** Let \(x_1 < x_2 \leq 3/4\) and \(x_1 + x_2 - 1 \geq 0\). Then \(K\) is a decreasing function of \(x_2\).

A solution specifies a pair of profits \((\Gamma_1, \Gamma_2)\) for each pair of locations, i.e., the ratio \((\Gamma_1 - \Gamma_{1N})/(\Gamma_2 - \Gamma_{2N})\) is given for each pair of locations. Suppose that \(x_1 < x_2 \leq 3/4\), \(x_1 + x_2 - 1 \geq 0\) and this ratio can be written as \((\Gamma_1 - \Gamma_{1N})/(\Gamma_2 - \Gamma_{2N}) = 1 + t(x_1 + x_2 - 1)J\).
Claim 12 Suppose that a solution satisfies (P1)–(P4). In addition assume that for any 1/4 ≤ 
1/2 ≤ 1/2 is an equilibrium then any pair of symmetric locations (y1, y2), x1 < y1 < 1/2 is also an 
equilibrium.

Proof Since (x1, x2) is an equilibrium, J(x1, x2) ≤ K(x1, x2) for any x1 ∈ (x1, x2), by 
arguments similar to those in the proofs of Theorems 1 and 2. In particular, for any y1 ∈ (y1, 
y2), J(y1, x2) ≤ K(y1, x2).

Since y2 < x2 and J(·, ·) is nondecreasing in its second argument, J(y1, y2) ≤ J(y1, x2).
By Claim 11, K(·, ·) is decreasing in its second argument. So, K(y1, x2) < K(y1, y2). Therefore, 
J(y1, y2) < K(y1, x2) for any y1 ∈ (y1, y2). So, (y1, y2) is an equilibrium.

The egalitarian solution serves as an illustration. Since (Γ1 – Γ1N)/(Γ2 – Γ2N) = 1, J(·, ·) 
is identically zero and the set of equilibria is convex.

In the example presented in this section, J(x1, x2) > 0 for 1/2 < x2 < 0.7 and J(x1, x2) 
= 0 for x2 > 0.7. It is not nondecreasing, hence, does not satisfy condition (ii) in Claim 12. 
As noted before, the set of equilibria in this example is not convex.

10.3 Proof of Claim 11

In this subsection, derivatives with respect to x2 are denoted by the prime symbol.

\[ \sigma' = \frac{t(1 - x_2)}{A - tx_1} \leq \frac{1}{2} \]

\[ g' = \frac{t(1 - 2\sigma')}{A - tx_1} \geq 0 \]

The numerator of \( \tilde{K} \) is \( 2 - (x_2 - x_1)[(7/3) + (1 + x_1 - x_2)g] \). Its derivative is

\[ -\left[ \frac{7}{3} + (1 + x_1 - x_2)g \right] - (x_2 - x_1)[-g + (1 + x_1 - x_2)g'] \]

\[ = -\left[ \frac{7}{3} + (1 + 2x_1 - 2x_2)g + (x_2 - x_1)(1 + x_1 - x_2)g' \right] \]

The denominator of \( \tilde{K} \) (ignoring 2) is \( [A - tx_1^2 + t(x_2 - x_1)(2\sigma - x_1 - x_2)](1 - \sigma) - \Gamma_{2N} \).

Its derivative is

\[ -\sigma'[A - tx_1^2 + t(x_2 - x_1)(4\sigma - x_1 - x_2 - 2)] + 2t(1 - \sigma)(\sigma - x_2) - \Gamma_{2N}^\prime \]
= -t(1 - x_2) - 2t(1 - \sigma)(x_2 - \sigma) - \Gamma_{2N}' - t(x_2 - x_1)\sigma'(4\sigma - x_1 - x_2 - 2)

Therefore, to show that \( \bar{K} \) is decreasing in \( x_2 \), one needs to show that

\[
- \left( \left[ A - tx_1^2 + t(x_2 - x_1)(2\sigma - x_1 - x_2) \right] (1 - \sigma) - \Gamma_{2N} \right) \\
\left[ \frac{7}{3} + (1 + 2x_1 - 2x_2)g + (x_2 - x_1)(1 + x_1 - x_2)g' \right] \\
+ \left( 2 - (x_2 - x_1) \left[ \frac{7}{3} + (1 + x_1 - x_2)g \right] \right) \\
\left[ t(1 - x_2) + 2t(1 - \sigma)(x_2 - \sigma) + \Gamma_{2N}' + t(x_2 - x_1)\sigma'(4\sigma - x_1 - x_2 - 2) \right] < 0
\]

Or, equivalently,

\[
\left( \left[ A - tx_1^2 + t(x_2 - x_1)(2\sigma - x_1 - x_2) \right] (1 - \sigma) - \Gamma_{2N} \right) \\
\left[ \frac{7}{3} + (1 + 2x_1 - 2x_2)g + (x_2 - x_1)(1 + x_1 - x_2)g' \right] \\
- \left( 2 - (x_2 - x_1) \left[ \frac{7}{3} + (1 + x_1 - x_2)g \right] \right) \\
\left[ t(1 - x_2) + 2t(1 - \sigma)(x_2 - \sigma) + \Gamma_{2N}' + t(x_2 - x_1)\sigma'(4\sigma - x_1 - x_2 - 2) \right] > 0
\]

1 + 2x_1 - 2x_2 \geq 0 since \( x_1 < x_2 \leq 3/4 \) and \( x_1 + x_2 - 1 \geq 0 \). 4\sigma - x_1 - x_2 - 2 \leq 0 since \( 2\sigma - x_1 - x_2 \leq 0 \) and \( \sigma \leq 1 \). Since \( 1/2 \leq \sigma < t(1 - x_2) + 2t(1 - \sigma)(x_2 - \sigma) \leq t(1 - x_2) + t(x_2 - \sigma) = t(1 - \sigma) \leq t/2 \). \( \Gamma_{2N} = t(x_2 - x_1)(4(x_1 - x_2)^2)/18 \). So, \( \Gamma_{2N}' = t(4(x_1 - x_2)(4x_1 - 3x_2)/18 \leq t/2 \) since \( 4 - x_1 - x_2 \leq 3 \) and \( 4 + x_1 - 3x_2 \leq 3 \).

Therefore, it suffices to show that

\[
[A - tx_1^2 + t(x_2 - x_1)(2\sigma - x_1 - x_2)](1 - \sigma) - \Gamma_{2N} > \frac{6}{7}t
\]

The LHS is a decreasing function of \( x_1 \). When \( x_1 = x_2 \), it is

\[
(A - tx_2^2) \left( 1 - \frac{A - t(x_1 - x_2)^2}{2(A - tx_2^2)} \right) = \frac{A - 2tx_2^2 + t(1 - x_2)^2}{2}
\]

Since \( A \geq 3t \) and \( x_2 \leq 3/4 \), this is greater than or equal to \((31/32)t\). 31/32 > 6/7, so the inequality holds.

This proves the claim.
11 Some Details Pertaining to Figure 3

Four claims are established below. (1) Point L lies above Point V in Figure 3. (2) Point L lies below line EG in Figure 3. (3) The function \((4 - x_1 - x_2)^2\hat{K}\) is increasing in \(x_1\). (4) The function \(h\) is increasing in \(x_1\).

11.1 Point L lies above Point V in Figure 3

For \(\hat{x}_1 \in [x_1, x_2]\), let

\[
H(\hat{x}_1) = 2(2 + \hat{x}_1 + x_2)^2\hat{p}_2(1 - \hat{z}) - (4 - \hat{x}_1 - x_2)\hat{p}_2(1 - \hat{z}) - (4 - \hat{x}_1 - x_2)^2(A - t\hat{x}_1^2)
\]

\[
= 2(2 + \hat{x}_1 + x_2)^2\hat{p}_2(\hat{x}_1) - (4 - \hat{x}_1 - x_2)^2(A - t\hat{x}_1^2)
\]

Derivatives with respect to \(\hat{x}_1\) are denoted by the prime symbol.

\[
\frac{H'}{2} = 2(2 + \hat{x}_1 + x_2)\hat{p}_2(2 + 2\hat{x}_1 + 2\hat{x}_2)\hat{p}_2(4 - \hat{x}_1 - x_2)(A - t\hat{x}_1^2)
\]

\[
\frac{H''}{2} = 2\hat{p}_2 + 2(2 + \hat{x}_1 + x_2)\hat{p}_2(2 + \hat{x}_1 + x_2)\hat{p}_2(2 + \hat{x}_1 + x_2)^2\hat{p}_2(4 - \hat{x}_1 - x_2)(A - t\hat{x}_1^2)
\]

Since \(\hat{p}_2 < (A - t\hat{x}_1^2)/2\), to show that \(H'' < 0\), it is enough to show that both \(\hat{p}_2\) and \(\hat{p}_2''\) are negative. Since \(\hat{p}_2 = A - t\hat{x}_1^2 + 2t(x_2 - \hat{x}_1)\hat{z}, \hat{z} = (A - t\hat{x}_1^2)/[2(A - t\hat{x}_1^2)]\) and \(\hat{p}_2 = \hat{p}_2(1 - \hat{z})\),

\[
\hat{p}_2' = -2t\hat{z} + 2t(x_2 - \hat{x}_1)\hat{z}'
\]

\[
\hat{z}' = \left[\frac{A - t\hat{x}_1^2}{2}\right] \frac{2t\hat{x}_1}{(A - t\hat{x}_1^2)^2} = \frac{2t\hat{x}_1}{A - t\hat{x}_1^2}
\]

\[
\hat{p}_2' = \hat{p}_2(1 - \hat{z}) - \hat{p}_2\hat{z}' = -[A - t\hat{x}_1^2 + 2t(x_2 - \hat{x}_1)(2\hat{z} - 1)]\hat{z}' - 2t\hat{z}(1 - \hat{z})
\]

Since \(\hat{z}' > 0\) and \(\hat{z} \geq 1/2, \hat{p}_2'' < 0,\)

\[
\hat{p}_2'' = -[A - t\hat{x}_1^2 + 2t(x_2 - \hat{x}_1)(2\hat{z} - 1)]\hat{z}''
\]

\[
-[-2t(2\hat{z} - 1) + 4t(x_2 - \hat{x}_1)\hat{z}']\hat{z}' + 2t(2\hat{z} - 1)\hat{z}'
\]

\[
= -[A - t\hat{x}_1^2 + 2t(x_2 - \hat{x}_1)(2\hat{z} - 1)]\hat{z}'' - 4t(x_2 - \hat{x}_1)(\hat{z}')^2 + 4t(2\hat{z} - 1)\hat{z}'
\]

Clearly,

\[
\hat{z}'' = \frac{2t\hat{x}_1}{A - t\hat{x}_1^2}\hat{z}' + \hat{z} \left(\frac{2t\hat{x}_1}{A - t\hat{x}_1^2}\right)'
\]

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Since \(2\ell \dot{x}_1/(A - t \dot{x}_1^2)\) is increasing in \(\dot{x}_1\), to show that \(\hat{\Gamma}''_2 < 0\), it is enough to show that

\[
\frac{[A - tx_2^2 + 2t(x_2 - \dot{x}_1)(2\dot{\xi} - 1)]2\ell\dot{x}_1}{A - \dot{x}_1^2} > 4t(2\dot{\xi} - 1)
\]

\[
[A - tx_2^2 + 2t(x_2 - \dot{x}_1)(2\dot{\xi} - 1)]\dot{x}_1 > 2(A - t\dot{x}_1^2)(2\dot{\xi} - 1)
\]

\[
= 2(A - t\dot{x}_1^2) \left[ \frac{A - tx_2^2}{A - t\dot{x}_1^2} - 1 \right]
\]

\[
= 2t(\dot{x}_1 + x_1)(\dot{x}_1 - x_1)
\]

Since \(A - tx_2^2 > 2t\) and \(\dot{x}_1 + x_1 \leq 1\), the inequality holds.

Therefore, \(H''\) is negative.

Clearly, if \(\dot{x}_1 = x_1\), then \(\dot{\xi} = 1/2\) and \(H(x_1) = 0\). It needs to be shown that \(H(x_2) > 0\).

\[
H(x_2) = 2(2 + \dot{x}_1 + x_2)^2\dot{p}_2(1 - \dot{\xi}) - (4 - \dot{x}_1 - x_2)^2(A - tx_1^2)
\]

\[
= 8(1 + x_2)^2\dot{p}_2(1 - \dot{\xi}) - 4(2 - x_2)^2(A - tx_1^2)
\]

\[
\frac{H(x_2)}{4} = 2(1 + x_2)^2(A - tx_2^2)(1 - \dot{\xi}) - (2 - x_2)^2(A - tx_1^2)
\]

\[
= 2(1 + x_2)^2(A - tx_2^2) - 2(1 + x_2)^2(A - tx_2^2)\dot{\xi} - (2 - x_2)^2(A - tx_1^2)
\]

\[
= 2(1 + x_2)^2(A - tx_2^2) - (1 + x_2)^2(A - tx_1^2) - (2 - x_2)^2(A - tx_1^2)
\]

\[
= 2(1 + x_2)^2(A - tx_1^2) - (1 + x_2)^2 + (2 - x_2)^2(2t(2x_2 - 1)]
\]

\[
= 3(2x_2 - 1)(A - tx_2^2) - [(1 + x_2)^2 + (2 - x_2)^2]t(2x_2 - 1)
\]

\[
\frac{H(x_2)}{4(2x_2 - 1)} = 3(A - tx_2^2) - [(1 + x_2)^2 + (2 - x_2)^2]t
\]

This is a decreasing function of \(x_2\). When \(x_2 = 3/4\), the RHS is

\[
3A - \frac{27 + 49 + 25}{16}t = 3A - \frac{101}{16}t
\]

Since \(A \geq 3t\), \(H(x_2) > 0\), for \(x_2 \leq 3/4\).

11.2 Point L lies below Line EG in Figure 3

One needs to show that \([A - tx_2^2]/2 - 2t(x_2 - \dot{x}_1)(\dot{x}_1 + x_2 - 1)/3\) is an increasing function of \(\dot{x}_1\). Its derivative with respect to \(\dot{x}_1\) is

\[
\frac{2t}{3}(2\dot{x}_1 - 1) + [A - tx_2^2 + 2t(x_2 - \dot{x}_1)(2\dot{\xi} - 1)]\dot{\xi}' + 2t \dot{\xi}(1 - \dot{\xi})
\]

Since \(\dot{\xi}'\) is positive, it suffices to show that \(2\dot{x}_1 - 1 + 3\dot{\xi}(1 - \dot{\xi})\) is positive. This follows from the facts that \(1/2 \leq \dot{\xi} < 2/3\) and \(\dot{x}_1 \geq 1/4\).
11.3 An Increasing Function

That \((4 - x_1 - x_2)^2 \tilde{K}\) is an increasing function of \(x_1\) can be seen as follows.

The proof of Claim 9 shows that the denominator of \(\tilde{K}\) is decreasing in \(x_1\), the numerator of \(\tilde{K}\) is less than 2 and its derivative is greater than 2. So, \((4 - x_1 - x_2)^2\) times the numerator of \(\tilde{K}\) is increasing in \(x_1\). Hence, \((4 - x_1 - x_2)^2 \tilde{K}\) is increasing in \(x_1\).

11.4 A Monotone Property at Symmetric Locations

Throughout this subsection symmetry is assumed and the properties of \(h(x_1)\) are examined.

\(h(x_1)\) is defined as \((t/12)(4 - x_1 - x_2)^2 \tilde{K}\) at symmetric locations. Therefore, \(h(x_1) = (3t/4) \tilde{K}\). From the definitions

\[
g = \frac{t(x_1 + x_2 + 1 - 2\sigma)}{A - tx_1^2}
\]

\[
\tilde{K} = \frac{2 - (x_2 - x_1) \left((7/3) + (1 + x_1 - x_2)g\right) - 2[A - tx_1^2 + t(x_2 - x_1)(2\sigma - (x_1 - x_2)(1 - \sigma) - 2\Gamma_{2N}]
\]

The denominator of \(\tilde{K}\) at symmetric locations is \(A - t(1 - x_1)^2\). At symmetric locations, \(g = t/(A - tx_1^2)\). Therefore, at symmetric locations the numerator of \(\tilde{K}\) is

\[
2 - (1 - 2x_1) \left[\frac{7}{3} + \frac{2tx_1}{A - tx_1^2}\right]
\]

Its derivative is

\[
2 \left[\frac{7}{3} + \frac{2tx_1}{A - tx_1^2}\right] - 2t(1 - 2x_1) \left[\frac{A - tx_1^2 + 2tx_1^2}{(A - tx_1^2)^2}\right]
\]

\[
= 2 \left[\frac{7}{3} + \frac{2tx_1}{A - tx_1^2}\right] - 2t(1 - 2x_1) \left[\frac{A + tx_1^2}{(A - tx_1^2)^2}\right]
\]

Since \((A + tx_1^2)/(A - tx_1^2)\) \(\leq 2\), \(1 - 2x_1 \leq 1/2\) and \(2t/(A - tx_1^2) \leq 1\), the derivative is greater than \(11/3\).

The denominator of \(\tilde{K}\) at symmetric locations is \(A - t(1 - x_1)^2\). Its derivative is \(2t(1 - x_1)\). The numerator of \(\tilde{K}\) is less than 2. Since \((11/3)[A - t(1 - x_1)^2] - 4t(1 - x_1) > 0\), \(\tilde{K}\) is an increasing function of \(x_1\) (at symmetric locations). Therefore, \(h\) is an increasing function of \(x_1\).

\(h(x_1) = (3t/4) \tilde{K} \leq (3t/2)/[A - t(1 - x_1)^2] \leq 3/4\).
The numerator of $\tilde{K}$ is

\[ 2 - (1 - 2x_1) \left[ \frac{7}{3} + \frac{2tx_1}{A - tx_1^2} \right] \]
\[ \geq 2 - (1 - 2x_1) \frac{7}{3} - (1 - 2x_1)x_1 \]
\[ \geq \frac{7}{4} - (1 - 2x_1) \frac{7}{3} \]

This is greater than or equal to $(7/4) - (7/6) = 7/12$. Therefore, $h(x_1) = (3t/4)\tilde{K} \geq (3t/4)(7/12)/[A - t(1 - x_1)^2] \geq (7/16)t/[A - t(1 - x_1)^2] \geq (7/16) (t/A) \geq (2/5) (t/A)$. 

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