ABSTRACT: THE CONSTRUCTION OF THE REAL NUMBER SYSTEM.

Currently, there is no modules in the mathematics department of NUS that offer a rigorous treatment of the construction of the real number system. The real number system is simply taken for granted, with all its associated properties. Although this current approach of assuming the existence of the real number system is unarguably reasonable and practical for most undergraduate mathematics, it does have several drawbacks.

Firstly, since the real number system is so fundamental to mathematics, at least at the undergraduate level, the omission of the treatment of its construction would seem to represent a knowledge gap in the average mathematics student’s training. Secondly, although most mathematics student may causally dismiss the real number system as intuitively obvious, there is cause for concern that this perspective was borne out of familiarity, and not intuition. For example, can anyone easily visualize an irrational number or give an intuitive argument for its existence? Why is it that the order completeness property holds in the real number system, or, from an axiomatic perspective, why does mathematicians ascribe to the real number system the order completeness property? When confronted with such questions, one may find that the real number system may not be so intuitive after all, even though it is such a familiar system.

On an intuitive level, the real number is certainly deeper than the natural numbers, integers or rationals. We can give physical analogy to the above 3 types of numbers easily, but can we also give a physical analogy to an irrational number? All of these sums up to one conclusion. The real number system is not intuitive at all. We do not know where it came from. Neither do we know the reason for its existence. Yet we are taught the mechanics of the real number system and have become so familiar with it that we never ever question its existence! For all we know, the real number system may turn out to be a ‘false’ system and this would make most of the mathematics that we learnt meaningless. There is hence a real need to question the existence of the real number system.

The aim of this project is to construct the real number system by starting with the axioms of Peano, which basically assume a counting system that modeled the natural number system. Most of the concepts employed in the construction are actually concepts taught at the undergraduate level, thereby demonstrating that it is viable to perhaps set up an undergraduate module to expound such a construction. Also, concepts such as integral domain, ring, field etc are actually borne out through a deep understanding of the various number systems, and so exposure to a rigorous construction of the real number system would enable the student to better appreciate the motivation behind such concepts. We will briefly outline the approach used in the construction and mention some of the associated concepts.

We start by assuming the axioms of Peano, which gives us the rules to create a natural number system. The construction proceed to show that such a natural number system is unique and that we can define the familiar addition, multiplication and exponentiation on it. Order is also defined and we then proceed to show that the system is well-ordered, order complete and that it possess the Archimedean Property.
The integer system is created from the natural number system by means of defining an equivalence relation on the set of pairs of natural numbers. The important conclusion here is that the integer system is an ordered integer domain and that it is also unique. We demonstrate several important concepts associated with integers and show that they are relevant also within our technical framework. In particular, we proved the fundamental theorem of Arithmetic. We also showed that the integer system is order complete but not well-ordered.

The rationals are also created from the integer system by means of defining a suitable equivalence relation. It is similarly shown to be an ordered field and that it is unique. Familiar results such as the denseness of rationals are also proved. This portion of the construction is significant, as we will argue intuitively that the rationals are inadequate and that an extension of the rational system is necessary. This is basically demonstrated with the drawing of a triangle with a side that is non-rational in length. The technical statement can be loosely represented by the theorem that say that the rational system is not order complete. We will also prove the seemingly unrelated fact that the rational system is not Cauchy complete, but the construction of the actual real number system will reveal that this two concepts are in fact inexorably linked.

For the construction of the actual real number system, we actually gives two such constructions. One was by using Dedekind’s cut and the other an approach by Georg Cantor using Cauchy sequences. Both started off with seemingly different assumption of what the real number system should be. Dedekind’s cut was used to extend the field of rational to an order complete field while Cantor’s approach aims to make it Cauchy complete instead. By following two approaches, we do not contradict our philosophy that we do not know what the real number system should be. We simply found two incompleteness of the rational system, and we proceed to complete each one separately to see what we obtain. The wonderful result is that both Dedekind and Cantor’s approach ended up with a common number system, and that this system is unique. We will then argue that this common number system is suitable to be the real number system, and is in fact the real number system that we have been using all along.

The main aim of the project is to demonstrate that the existence of the real number system is not that simple and obvious but a rigorous verification on the part of the student is not impossible. We choose to start with the axioms of Peano since that, in the author’s opinion, spelt out a natural system that can be intuitively accepted. At the very least, it is more intuitively clear than the axiom that the real number system must be order complete!