Elements of Finite Order of $GL_4(\mathbb{Z})$

Carolina A$^1$ and Lang M.L$^2$

Department of Mathematics, National University of Singapore
2 Science Drive 2, Singapore 117543

ABSTRACT
The purpose of the report is to determine a complete list of maximal elementary abelian 2-groups of $GL_4(\mathbb{Z})$ up to conjugation. In order to achieve this, one needs to determine the conjugacy classes of matrices of order 2 of $GL_4(\mathbb{Z})$, $GL_3(\mathbb{Z})$, and $GL_2(\mathbb{Z})$. These conjugacy classes are given in the following.

Involutions of $GL_4(\mathbb{Z})$, $GL_3(\mathbb{Z})$, and $GL_2(\mathbb{Z})$.

By [Y], $GL_4(\mathbb{Z})$ possesses 8 conjugacy classes. Further, an element of order 2 of $GL_4(\mathbb{Z})$ is conjugate to one of the following:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

$^1$Student
$^2$Associate Professor
Denote by $\Omega_4$ the set of the above matrices.

By [T], $GL_3(\mathbb{Z})$ possesses 5 conjugacy classes. Further, an element of order 2 of $GL_3(\mathbb{Z})$ is conjugate to one of the following:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$-\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Denote by $\Omega_3$ the set of the above matrices.

By [T], $GL_2(\mathbb{Z})$ possesses 6 conjugacy classes. Further, an element of order 2 of $GL_2(\mathbb{Z})$ is conjugate to one of the following:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Denote by $\Omega_2$ the set of the above matrices.

Applying a well known result of Linear Algebra, the order of a maximal abelian elementary 2-group of $GL_4(\mathbb{Z})$ is 16. Suppose that

$$\langle x, y, z, w \rangle$$
is a maximal abelian elementary 2-group of $GL_4(\mathbb{Z})$. Since $x$ is of order 2, without loss of generality, we may assume that $x \in \Omega_4$. In the case

$$x = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

since $x$ is in the center of $GL_4(\mathbb{Z})$, one may assume further that $y \in GL_4(\mathbb{Z})$. Since $y \neq \pm I_{4 \times 4}$, $y$ is not in the center of $GL_4(\mathbb{Z})$. It is an easy matter to determine the centralizer

$$C(y) = C_{GL_4(\mathbb{Z})}(y)$$

of $y$ in $GL_4(\mathbb{Z})$. A case by case study shows that elements in $C(y)$ can be decomposed into block of matrices of the following form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \text{ or } \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

This implies that $z$ is of the above form. Since $A$ and $B$ are either $3 \times 3$, $2 \times 2$ or $1 \times 1$ and $z \in C(y)$ is of order 2, it follows that $A, B = \pm 1$ or $A$ and $B$ are conjugate to the matrices in $\Omega_3 \cup \Omega_2$. Let $\tau$ be an integral basis of $\mathbb{R}^4$ such that

$$[z]_{\tau} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \text{ or } \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix},$$

where $A, B \in \Omega_3 \cup \Omega_2$. One can choose the basis $\tau$ appropriately so that

(i) $\det P = \pm 1$, where $P$ is the matrix associated to $\tau$,

(ii) $P^{-1}xP = x$, $P^{-1}yP = y$. 
As a consequence, the set of conjugacy classes of elementary abelian 2-groups of order 8 such that

\[ x = \begin{pmatrix}
  -1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & -1 & 0 \\
  0 & 0 & 0 & -1 \\
\end{pmatrix} \]

can be determined. As this procedure is applicable to \( w \) as well, the set of conjugacy classes of elementary abelian 2-groups of order 16 can be determined.

In the case

\[ x \neq \begin{pmatrix}
  -1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & -1 & 0 \\
  0 & 0 & 0 & -1 \\
\end{pmatrix}, \]

It is clear that our methodology works as well. Consequently, the set of conjugacy classes of elementary abelian 2-groups of order 16 can be determined.

A complete list of maximal elementary abelian 2-group can be determined according. Such a list can be found in the Appendix A of the report.

REFERENCES
