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Modular Forms and Gravitational Anomaly Cancellation in String Theory

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String theory is a candidate theory in particle physics for reconciling quantum mechanics and general relativity, i.e. quantum gravity. Moreover, it's also a candidate of TOE ("theory of everything"), describing all the known fundamental forces and matters in our universe in a mathematically complete system. In conventional theories, elementary particles are mathematical points, whereas, the fundamental objects in string theory are 1-dimensional oscillating lines or loops.

"As is well known, in the early 1980s, it appeared that superstrings could not describe parity violating theories, because of quantum mechanical inconsistencies due to anomalies. The discovery that in certain cases the anomaly could cancel was important for convincing many theorists that string theory is a promising approach to unification.", the famous string theorist John H. Schwarz remarked in [10].

In an important paper of Alvarez-Gaumé and Witten [3], it is shown that in certain parity-violating gravity theory in 4k+2 dimensions, when Weyl fermions of spin-1/2 or spin-3/2 or self-dual antisymmetric tensor field are coupled to gravity, perturbative anomalies $I_{1/2}$, $I_{3/2}$ and I_A occur and moreover the authors show that there are cancellation formulas for these anomalies in dimensions 2, 6, 10. More precisely, in dimensions 2, 6, 10, the following anomaly cancellation formulas hold respectively,

$$-I_{1/2}+I_A=0 \qquad (0.1)$$

$$21I_{1/2}-I_{3/2}+8I_A=0 \qquad (0.2)$$

and $$-I_{1/2}+I_{3/2}+I_A=0 \qquad (0.3)$$

Type IIB superstring theory is such an anomaly-free ten-dimensional parity-violating theory. There are five types of superstring theories: I, IIA, IIB, HE and HO.

The mathematical understanding of the above anomalies relates to the Atiyah-Singer index theory [1, 2]. The Atiyah-Singer index theory is one of the greatest achievements in mathematics of the 20th century. This theory establishes connections among analysis, topology and geometry on differential manifold (curved space that locally looks like Euclidean space). In mathematical language, the above anomaly cancellation problem is the following. Consider the fiber bundle Z→M→Y, where a family of spin manifold Z is parametrized by another connected manifold Y and the family as a whole forms a manifold M. In the string theory situation, Z is a 4k+2 dimensional spin manifold and Y is the quotient space of the space of metrics on Z by the action of certain subgroup of diffeomorphism group of Z. Let D^Z be the family of Dirac operators and D^Z_{sig} be the family of signature operators on the fiber bundle respectively. D^Z and D^Z_{sig} are first order elliptic operators. Let TZ be the bundle of vertical tangent spaces on M and T Z be its complexification. Then

$$I_{1/2}=R^{L_{D^Z}},$$

$$I_{3/2}=R^{L_{D^Z\otimes T Z}}-R^{L_{D^Z}},$$

$$I_A=-\frac{1}{8}R^{L_{D^Z_{sig}}},$$

where L represents certain line bundle, called the determinant line bundle, on the base Y associated to the elliptic operators and R represents the curvature of the determinant line bundle. Therefore (0.1)-(0.3) become

$$R^{L_{D^Z_{sig}}}+8R^{L_{D^Z}}=0 \qquad (0.4)$$

$$R^{L_{D^Z_{sig}}}+R^{L_{D^Z\otimes T Z}}-22R^{L_{D^Z}}=0 \qquad (0.5)$$

and $$R^{L_{D^Z_{sig}}}-8R^{L_{D^Z\otimes T Z}}+16R^{L_{D^Z}}=0 \qquad (0.6)$$

We see that the above anomaly cancellations are just cancellations of the curvatures of determinant line bundles of certain elliptic operators.

We find in [6] that formulas (0.4)-(0.6) are related to modular forms in number theory and by using modular forms we can generalize the above cancellation formulas to general 4k+2 dimension. A modular form is a holomorphic function on the upper half complex plane, which possesses certain symmetry with respect to the Möbius transformation $\tau\rightarrow\frac{a\tau+b}{c\tau+d}$. Modular forms are very important in arithmetics and the generalizations of modular forms are automorphic forms. It is still somewhat mysterious to see that anomaly cancellation problems in string theory are related to modular forms.

Let $q = e^{2\pi\sqrt{-1}\tau}$ One can actually show that $R^{L_{D^Z_{sig}}}$ is a piece in the q-series of a modular form $P_1(\tau)$ while $R^{L_{D^Z\otimes TZ}}$ as well as $R^{L_{D^Z}}$ are pieces in the q-series of another modular form $P_2(\tau)$ and moreover $P_1(-\frac{1}{\tau}) = (2\tau)^{2k+2}P_2(\tau)$. Then $R^{L_{D^Z_{sig}}}$ and $R^{L_{D^Z\otimes TZ}}$, $R^{L_{D^Z}}$ are related because $P_1(\tau)$ and $P_2(\tau)$are modularly related, i.e. there is some hidden symmetry. This gives us the generalizations of the anomaly cancellation formulas (0.4)-(0.6):

if Z is 8m+2 dimensional,

$$R^{L_{D^Z_{sig}}} - 8\sum_{r=0}^{m} 2^{6m-6r} R^{L_{D^Z\otimes b_r(TZ)}} = 0 \qquad (0.7)$$

if Z is 8m-2 dimensional,

$$R^{L_{D^Z_{sig}}} - \sum_{r=0}^{m} 2^{6m-6r} R^{L_{D^Z\otimes z_r(TZ)}} = 0 \qquad (0.8)$$

The b_r and z_r are certain operations of the tangent bundle determined by the q-series:

$$-\frac{1}{8} - 3\sum_{n=1}^{\infty}\sum_{\substack{d|n \\ d\ odd}} dq^{n/2}, \quad \sum_{n=1}^{\infty}\sum_{\substack{d|n \\ n/d\ odd}} d^3 q^{n/2}.$$

The method of using modular forms also produces more interesting results. For example, in 11 dimension, we can obtain the following result:

$$\exp\{2\pi\sqrt{-1}(\bar{\eta}(B^Z_{sig}) - 8\bar{\eta}(D^Z \otimes TZ) + 24\bar{\eta}(D^Z))\}$$

is a constant function on Y, where $\bar{\eta}$ is an important spectral function for Dirac operators. We would like to remark that 11 is a special dimension. In the second revolution of string theory, people realized that the five superstring theories are special cases a more fundamental theory: M-theory. 11 is the dimension the M-theory is modeled on. ■

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