NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

Ph.D. QUALIFYING EXAMINATION

ALGEBRA (SAMPLE PAPER)

Time allowed : 3 hours

Answer All Questions

1. [20 marks] Let $\varphi : M' \to M$ be a homomorphism of abelian groups.

   (a) Suppose that $\alpha : L \to M'$ is a homomorphism of abelian groups such that $\varphi \circ \alpha = 0$ (an example is the inclusion $\mu : \mathrm{Ker}\varphi \hookrightarrow M'$). Prove or disprove each of the following.

   (i) There is a unique homomorphism $\alpha_0 : \mathrm{Ker}\varphi \to L$ such that $\mu = \alpha \circ \alpha_0$.

   (ii) There is a unique homomorphism $\alpha_1 : L \to \mathrm{Ker}\varphi$ such that $\alpha = \mu \circ \alpha_1$.

   (b) Show that there is a four-term exact sequence

   $$0 \longrightarrow \mathrm{Ker}\varphi \xrightarrow{\mu} M' \xrightarrow{\varphi} M \xrightarrow{\epsilon} \mathrm{Coker}\varphi \longrightarrow 0.$$

   What is Coker φ?

   (c) Dualize (a); that is, find and prove the corresponding statement about Coker φ.

2. [15 marks] Let

   $$0 \longrightarrow M' \xrightarrow{\mu} M \xrightarrow{\epsilon} M'' \longrightarrow 0$$

   be a short exact sequence of R-modules, with M' finitely generated. Show that M is finitely generated if and only if M'' is. Give an example with M finitely generated but M' not finitely generated.

3. [15 marks] Show that an A-module P is projective if and only if for any short exact sequence

   $$0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$$

   of A-module homomorphisms, the corresponding sequence

   $$0 \longrightarrow \mathrm{Hom}_A(P, N') \xrightarrow{\mathrm{Hom}_a(P,f)} \mathrm{Hom}_A(P, N) \xrightarrow{\mathrm{Hom}_A(P,g)} \mathrm{Hom}_A(P, N'') \longrightarrow 0$$

   is exact.

4. [15 marks] Let V be a finite dimensional k-vector space, of dimension n, and let $\varphi : V \to V$ be an invertible k-linear transformation from V to itself. Let $\mathcal{B} = (e_1, ..., e_n)$ be a basis of V.

   (a) Show that

   $$\varphi(\mathcal{B}) := (\varphi(e_1), ..., \varphi(e_n)) \qquad \text{is also a basis of } V.$$

   (b) Let $A_\varphi \in M_n(k)$ be the matrix of φ with respect to $\mathcal{B}$ and $\mathcal{B}$. In terms of A_φ, compute

   (i) the matrix of φ with respect to $\mathcal{B}$ and $\varphi(\mathcal{B})$;

   (ii) the matrix of φ with respect to $\varphi(\mathcal{B})$ to $\varphi(\mathcal{B})$;

   (iii) the matrix of φ with respect to $\varphi(\mathcal{B})$ to $\mathcal{B}$.

   Justify your answers with proofs.

5. [15 marks] For each of the following statements about field extensions $F \subseteq K \subseteq L$, either prove or give a counterexample.

   (i) If L is a splitting field for a polynomial over F, then L is a splitting field for a polynomial over K.

   (ii) If L is a splitting field for a polynomial over K, and K is a splitting field for a polynomial over F, then L is a splitting field for a polynomial over F.

   (iii) If L is a splitting field for a polynomial over F, then K is a splitting field for a polynomial over F.

6. [10 marks] Find $\mathrm{Gal}(F(\zeta)/F)$ where $\zeta = \cos(2\pi/n) + i\sin(2\pi/n)$, $F = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, and $n = 4, 6, 12$ or a prime p.

7. [10 marks] Let $\mathfrak{S}$ denote the category whose objects are sets and whose morphisms are the functions from sets to sets. Let $P : \mathfrak{S} \to \mathfrak{S}$ be the functor that assigns to each set X its power set $P(X)$ (i.e. the set of all subsets of X), and assigns to each morphism $f : A \to B$ a corresponding morphism $P(f) : P(B) \to P(A)$ given by

   $$P(f)(X) = f^{-1}(X) \quad \text{for each } X \subset B.$$

   Show that P is a representable contravariant functor.