NUS National University of Singapore | College of Humanities and Sciences

CHS-FOS OPEN HOUSE 2023

13 MAY 2023

OPEN HOUSE 13TH MAY 2023

Introduction to Quantitative Finance

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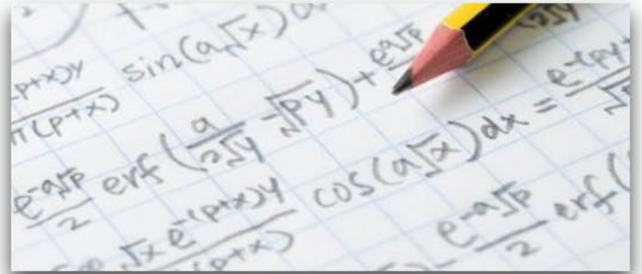
Department of Mathematics

A Door that Opens to Many Possibilities

Data Scientist

Statistician



Mathematician

Actuary

Operation Research Analyst

Information Security Analyst

Soft Engineer

University Professor

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What will you learn in QF?

- Interest Theory
- Programming to Implement Techniques of QF
- Mathematical Models of Finance
- Investment Portfolio Construction and Optimization
- Hedging and Risk
- Derivatives: Forwards, Futures, and Options
- Etc.

What do you need to know to learn QF?

- H2 math
- Graphs and transformations.
- Probability
- Calculus
- Linear Algebra
- Modeling and Programming
- Time Series Analysis
- Stochastic Calculus
- etc

A first example: the Mathematics of Retirement Planning

Power of Compounding

"Compound interest is the

eighth wonder of the world

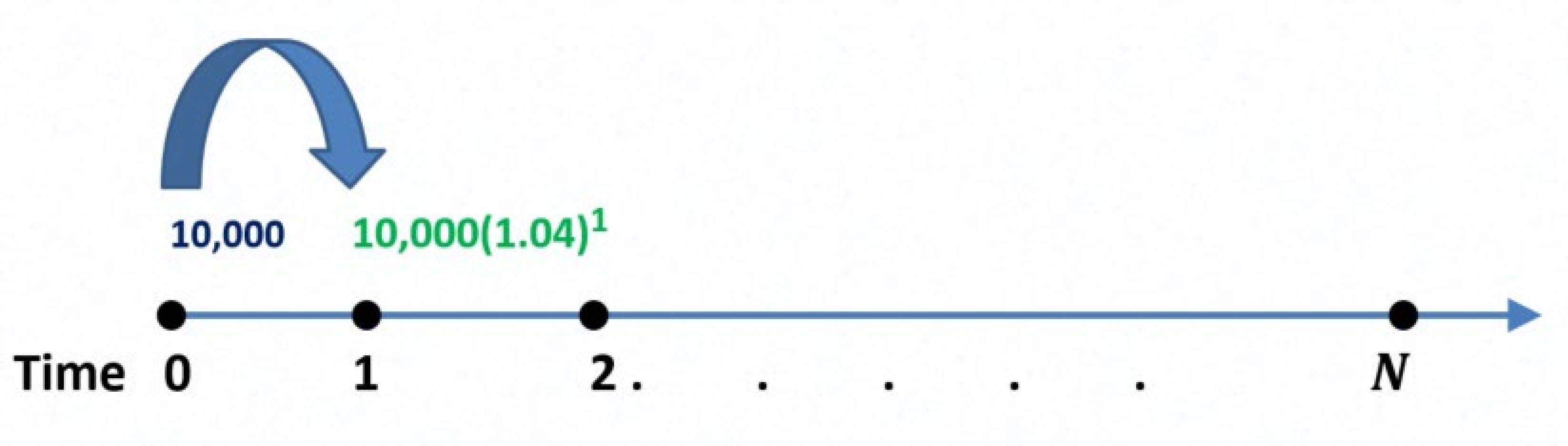
He who understands it, earns it …

he who doesn't … pays it "

– Albert Einstein

Power of Compounding

- Deposit $\$10,000$ at time t = 0.
- Interest of 4% is paid at the end of each period



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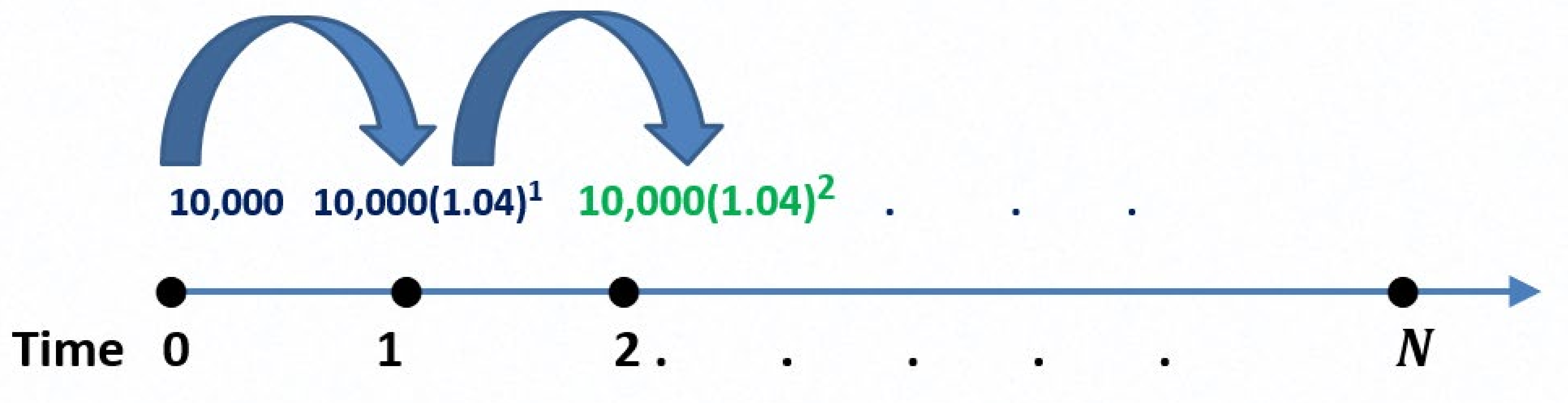

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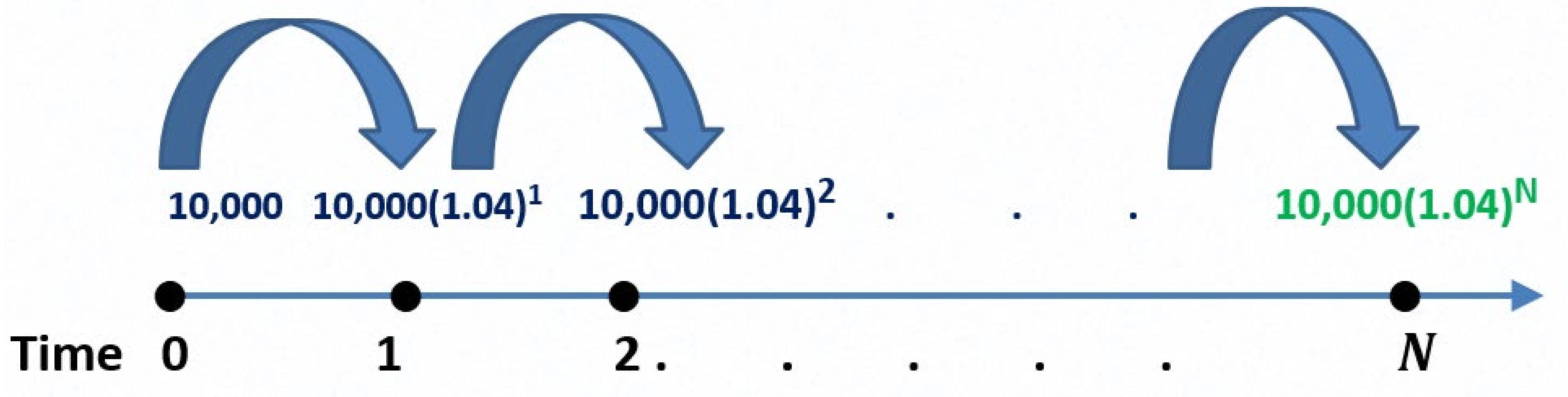

Power of Compounding

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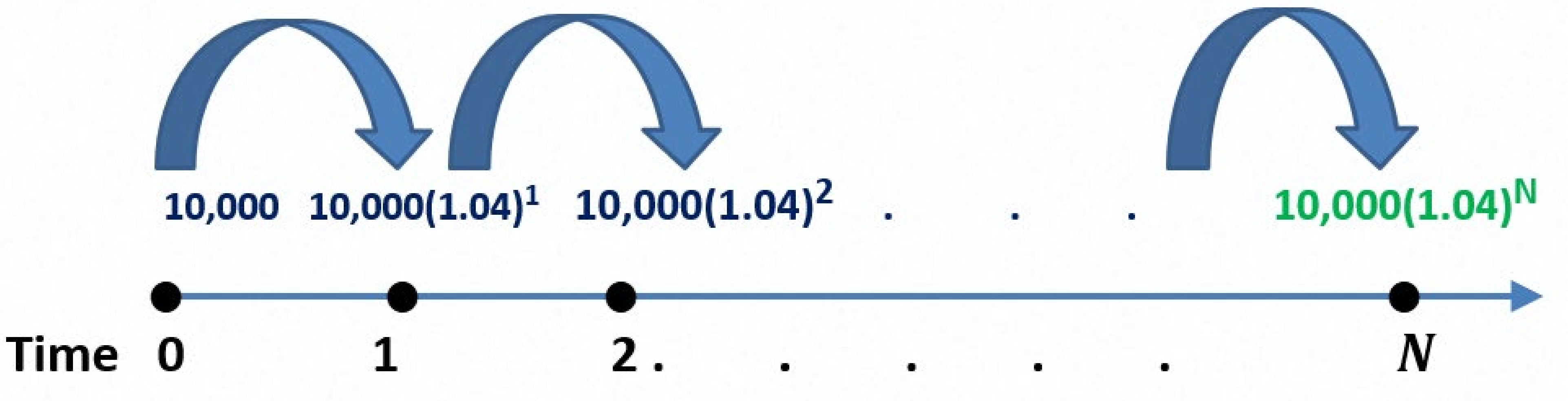
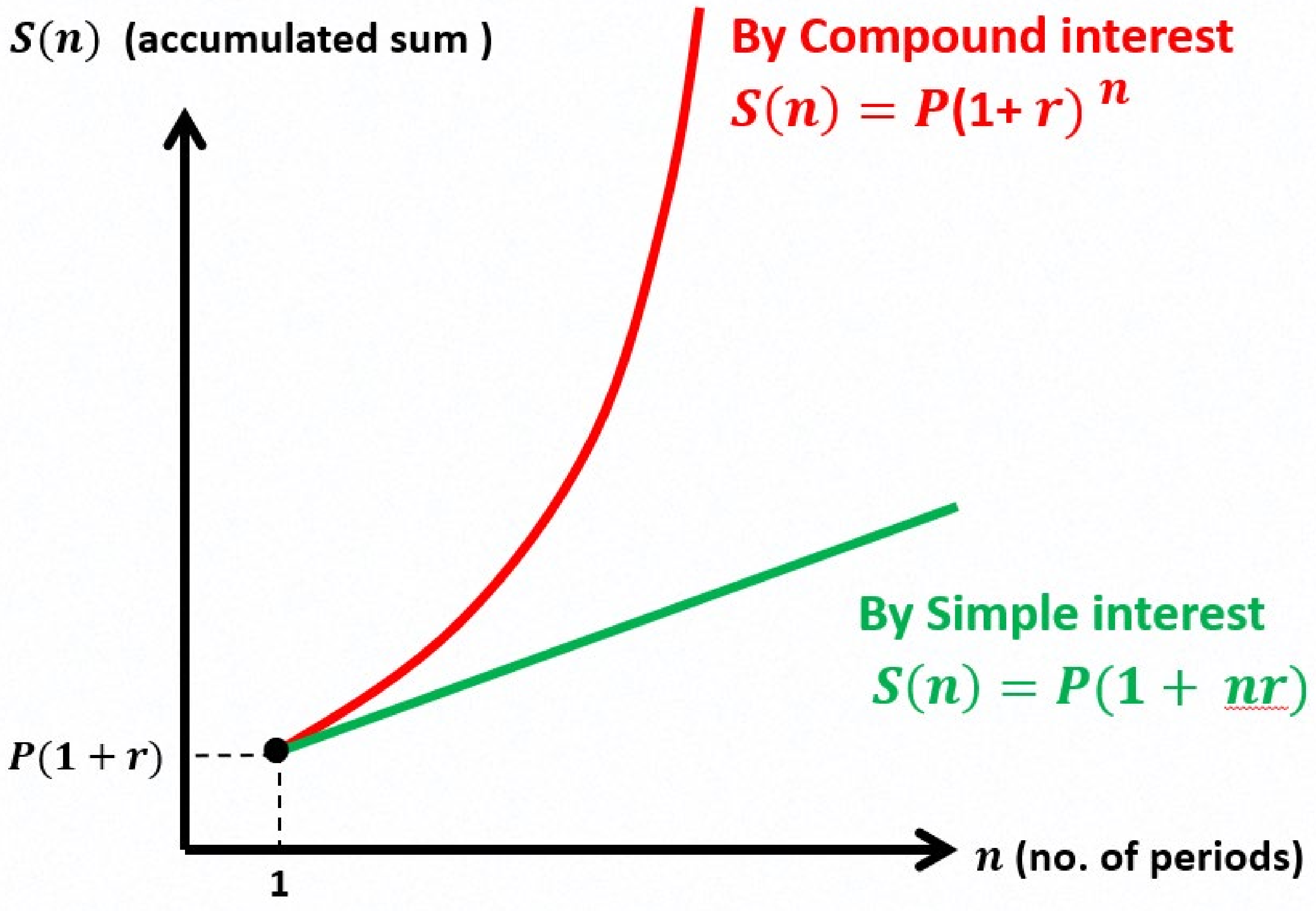


- $N = 40$
- $10,000(1.04)^{40} = 48,010$
- Interest earned = 380% !!

Power of Compounding



Cents & Sensibility

Q. How to save for retirement ?

Cents & Sensibility

Cents & Sensibility

Assumptions

- **You start working at age 25.**
- **Your desired retirement age is 65**
- **You wish to accumulate a retirement fund of**
- **$ 1 million**
- **You save a fixed amount annually**

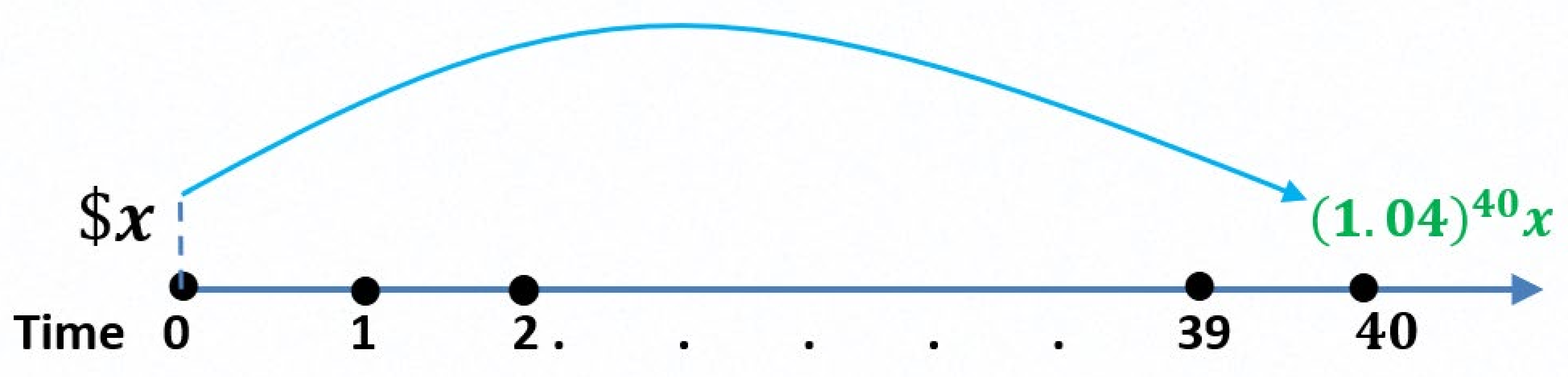
Cents & Sensibility

Plan

Save $\$x$ regularly at $t = 0, 1, 2, \ldots\ldots, 39$

(retire at $t = 40$)

Assuming interest of 4% is paid at the end of each period



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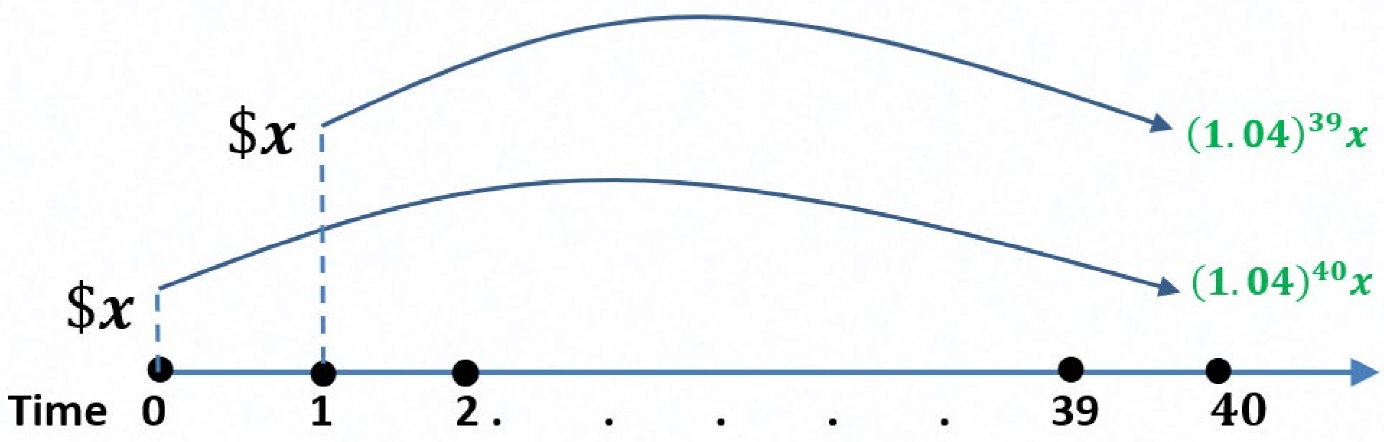
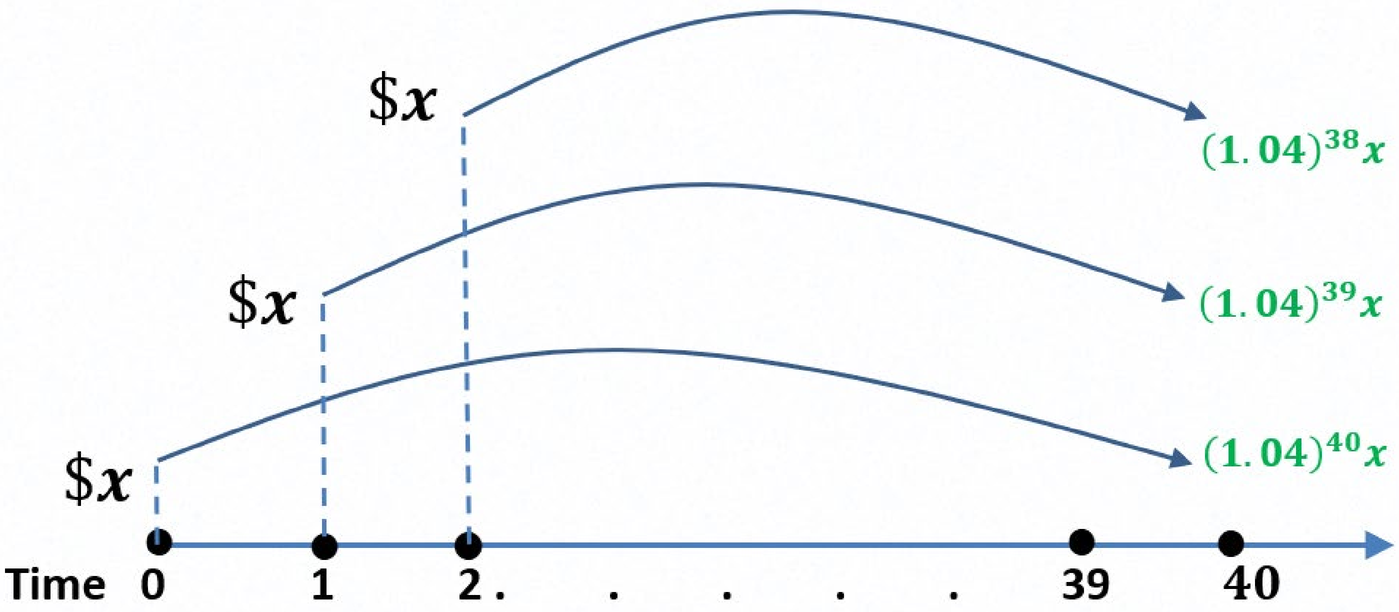


Cents & Sensibility



Cents & Sensibility

Accumulated sum at $t = 40$

$$= (1.04)^{40}x + (1.04)^{39}x + (1.04)^{38}x + \cdots . + (1.04)^{1}x$$

Cents & Sensibility

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Cents & Sensibility

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Geometric Series

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}$$

Cents & Sensibility

Accumulated sum at $t = 40$

$$= 1.04\left(\frac{1.04^{40} - 1}{0.04}\right)x$$

$$= 98.83x$$

Cents & Sensibility

To be a millionaire,

$$98.83x \geq 10^6$$

$$x \geq 10119$$

Save at least $10,119 annually for 40 years

You put in $\$\ 10119 \times 40 = \$404{,}760$

and earn an interest of $\$\ 595{,}240$ **!**

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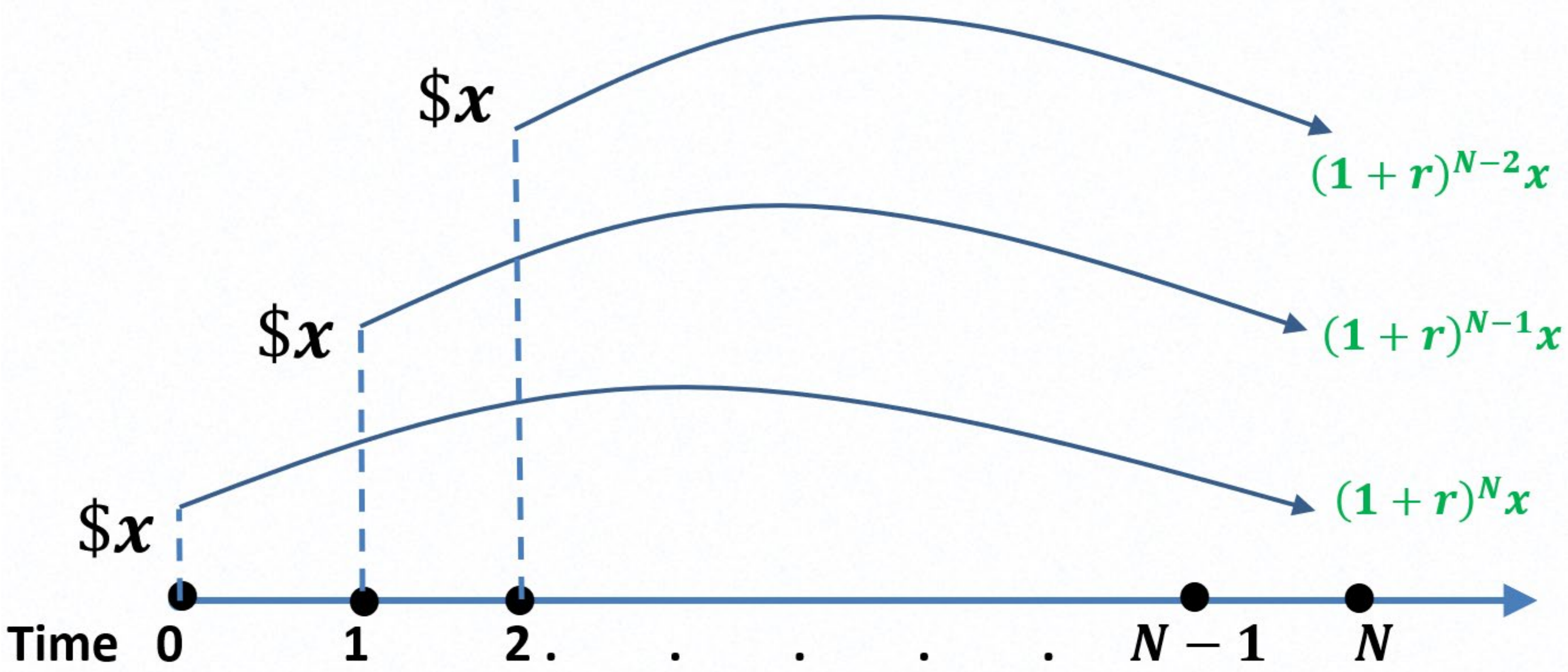
Cents & Sensibility

In general, if

you save $\$x$ regularly at $t = 0, 1, 2, \ldots\ldots, N$

rate of growth of your savings is r compounded annually



Cents & Sensibility

Total accumulated sum at $t = N$

$$S = (1+r)^N x + (1+r)^{N-1} x + (1+r)^{N-2} x + \cdots . + (1+r)^1 x$$

$$S = (1+r)\left(\frac{(1+r)^N - 1}{r}\right) x$$

Cents & Sensibility

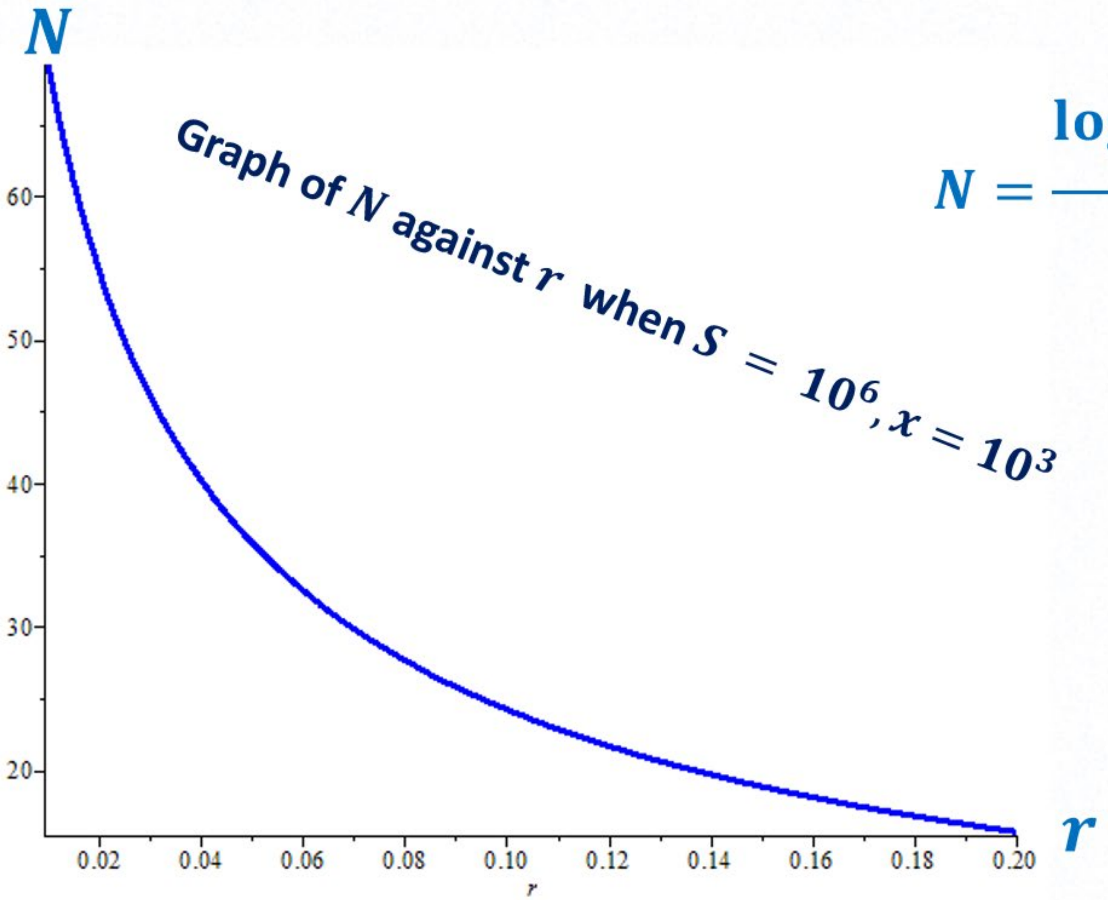
$$S = (1+r)\left(\frac{(1+r)^N - 1}{r}\right)x$$

- Inverse Relation between N & r

when x and S are fixed)

$$N = \frac{\log\left(\frac{rS}{x(1+r)} + 1\right)}{\log(1+r)}$$

Cents & Sensibility



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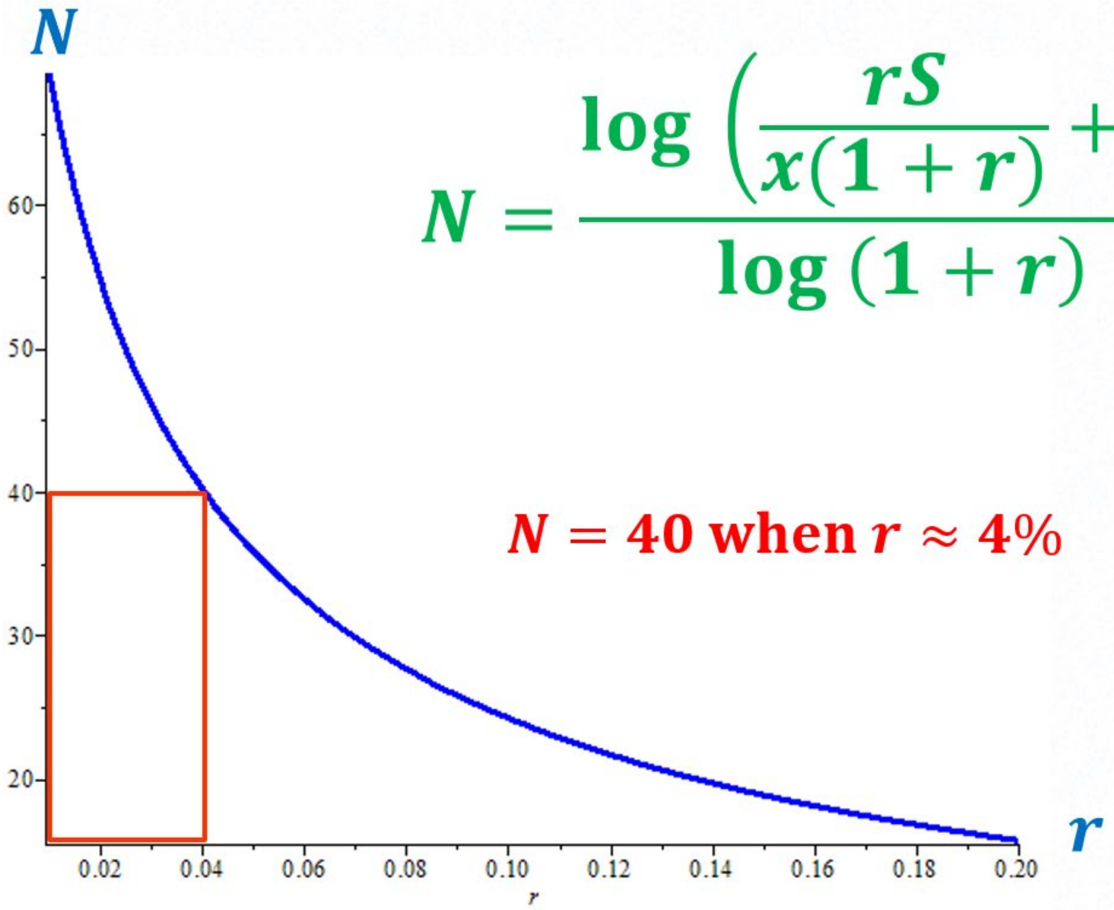
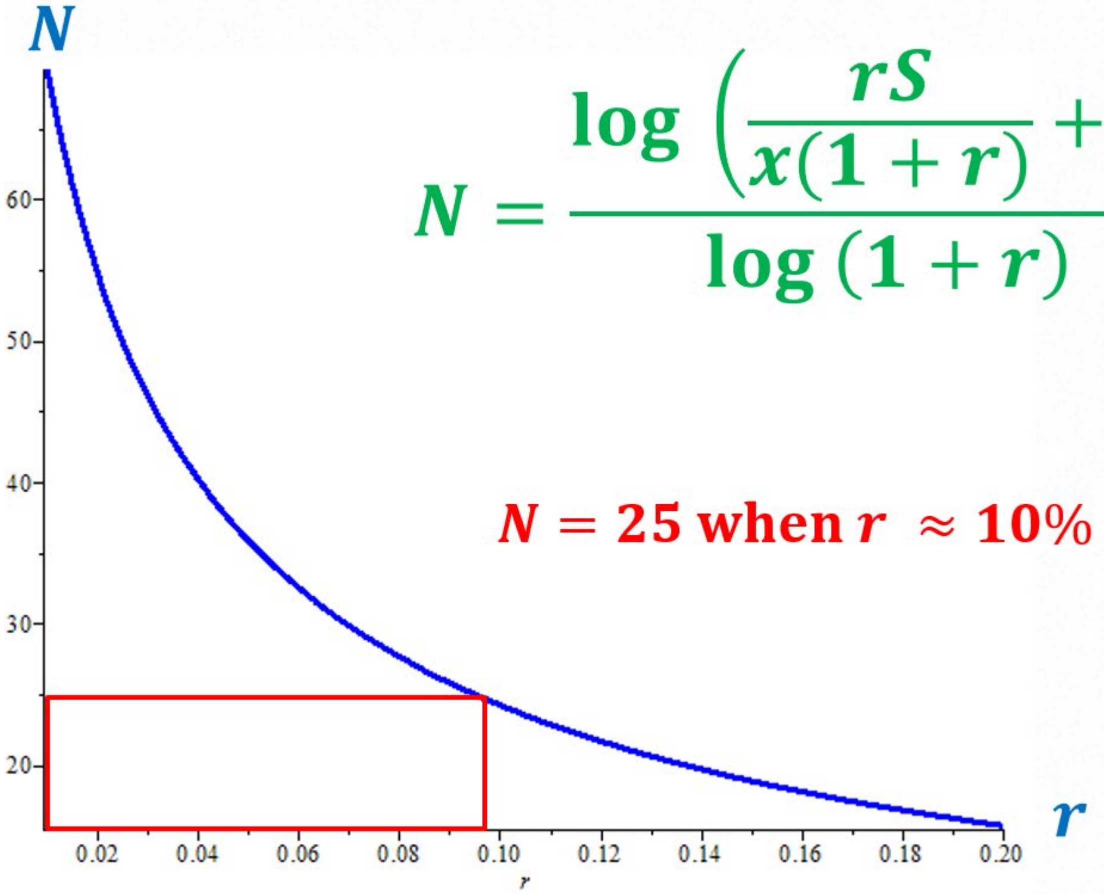
Cents & Sensibility



$$N = \frac{\log\left(\frac{rS}{x(1+r)} + 1\right)}{\log(1+r)}$$

$N = 40$ when $r \approx 4\%$

Cents & Sensibility



$$N = \frac{\log\left(\frac{rS}{x(1+r)} + 1\right)}{\log(1+r)}$$

$N = 25$ when $r \approx 10\%$

Cents & Sensibility

Cents & Sensibility

A More Realistic Plan

- You save more as you earn more

You save

$\$x$ at $t = 0$

$\$x \times (1 + a)$ at $t = 1$

$\$x \times (1 + a)^2$ at $t = 2$

....

Cents & Sensibility

A More Realistic Plan

- You spend more as you earn more

You spend

$\$y$ at $t = 0$

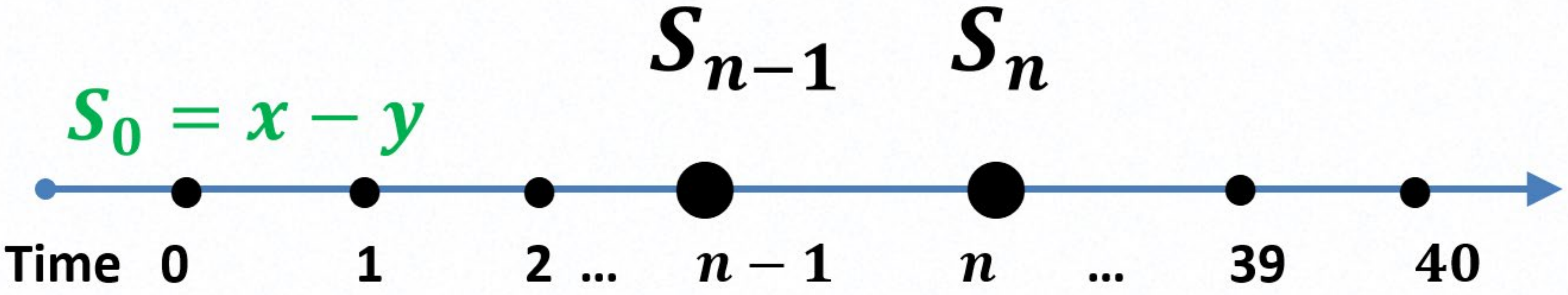
$\$y \times (1 + b)$ at $t = 1$

$\$y \times (1 + b)^2$ at $t = 2$

....

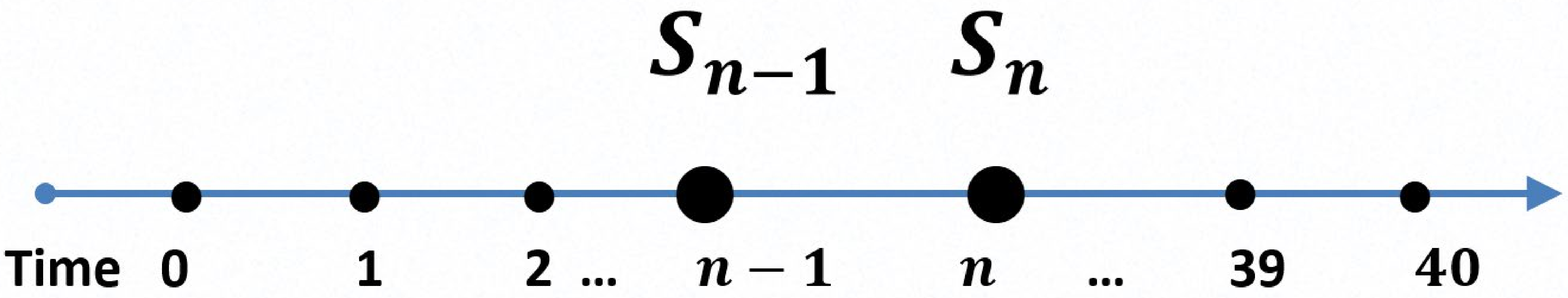
Cents & Sensibility

Let S_n = accumulated sum at time n



We would like to obtain a formula for S_n

Cents & Sensibility



Relation between S_{n-1} & S_n ?

Cents & Sensibility

S_{n-1} earns interest rS_{n-1} **INTEREST**

$\$x \times (1+a)^n$ is deposited **SAVE**

$\$y \times (1+b)^n$ is debited **SPEND**

Cents & Sensibility

First order difference equation (recurrence relation)

$$S_n = (1 + r)S_{n-1} + x(1 + a)^n - y(1 + b)^n$$

Interest | Saving | Spending

Cents & Sensibility

Solution is

$$S_n = (1+r)^n \left(x \sum_{k=0}^{n} \left(\frac{1+a}{1+r} \right)^k - y \sum_{k=0}^{n} \left(\frac{1+b}{1+r} \right)^k \right)$$

Difference of Two Geometric Series

Cents & Sensibility

EXAMPLE A

- **Save $20000 , increasing at annual rate of 5%**
- **Spend $10000, increasing at annual rate of 3%**
- **Interest rate = 4%**

Retirement fund in 40 years =

$$\$\ (1.04)^{40}\left(20000\sum_{k=0}^{40}\left(\frac{1.05}{1.04}\right)^{k} - 10000\sum_{k=0}^{40}\left(\frac{1.03}{1.04}\right)^{k}\right)$$

= $ 3, 164, 691

Cents & Sensibility

EXAMPLE B

- Save $20000 , increasing at annual rate of 5%
- Spend $20000, increasing at annual rate of 3%
- Interest rate = 4%

Retirement fund in 40 years =

$$\$\,(1.04)^{40}\left(20000\sum_{k=0}^{40}\left(\frac{1.05}{1.04}\right)^{k} - 20000\sum_{k=0}^{40}\left(\frac{1.03}{1.04}\right)^{k}\right)$$

= $ 1, 531, 528

Cents & Sensibility

EXAMPLE B

- Save $20000 , increasing at annual rate of 5%
- Spend $25000, increasing at annual rate of 3%
- Interest rate = 4%

Retirement fund in 40 years =

$$\$\ (1.04)^{40}\left(20000\sum_{k=0}^{40}\left(\frac{1.05}{1.04}\right)^{k}-25000\sum_{k=0}^{40}\left(\frac{1.03}{1.04}\right)^{k}\right)$$

= $ 714, 947

Cents & Sensibility

In a Nutshell

- **Start** planning
- **Save** regularly
- **Spend** prudently

Thank you.