

Department of Mathematics Faculty of Science

Research Highlight: Product structure and regularity theorem for totally nonnegative flag varieties

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The theory of total positivity is deeply connected to combinatorics, geometry, and representation theory, ranging from cluster algebras to the physics of scattering amplitudes.

An invertible matrix is called totally positive (resp. totally nonnegative) if all its minors are positive (resp. nonnegative). The study of such matrices dates back to Schoenberg and Grantmacher-Krein in 1930s. The theory has been generalized by Lusztig (Progr. in Math. 1994) to arbitrary split reductive groups. Lusztig further defined a totally nonnegative flag variety, which he described as a ``remarkable polyhedral subspace" of the usual flag variety. The geometry of such totally nonnegative flag varieties has been central to the recent research of amplituhedra, geometric objects introduced by physicists Arkani-Hamed and Trnka (J. High Energy Phys. 2014) underling the physics of scattering amplitudes.

We (Invent. Math. 2024) showed that the totally nonnegative part of a Kac-Moody flag variety (or rather any closed Richardson variety) is a regular CW complex homeomorphic to a closed ball. This generalizes the results by Galashin-Karp-Lam (JAMS 2021) and Hersh (Invent. Math. 2014).

References:

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