## Algebra Qualifying Exam August 2024

Please attempt all problems. Each problem is worth 10 points, except problems 6 and 9 which are each worth 15 points. The total possible score is 100 points.

## Problems

- 1. Classify all groups of order 14 up to isomorphism.
- 2. Compute the Galois group of the polynomial  $x^4 + 7$  over **Q**, and draw the lattice of all subfields of its splitting field.
- 3. Compute the group  $\operatorname{Ext}^{1}_{\mathbf{Z}}(\mathbf{Z}/10\mathbf{Z}, \mathbf{Z}/12\mathbf{Z})$ .
- 4. Let A be a Noetherian local ring. Prove that a finitely generated projective A-module is free.
- 5. Prove that if A is a Noetherian ring and M is a finitely generated A-module, then any A-submodule  $N \subset M$  is finitely generated.
- 6. i. Give the precise definition of a unique factorization domain.ii. Give the precise definition of an integrally closed domain.iii. Give the precise statement of the Hilbert basis theorem.
- 7. Determine the number of irreducible complex representations of the group  $S_4$ , along with their dimensions.
- 8. Classify all abelian groups of order 360 up to isomorphism.
- 9. Give one example of each of the following:
  - i. A commutative ring with exactly two prime ideals.
  - ii. An integral domain which is not a principal ideal domain.
  - iii. An algebraically closed field with countably many elements.