Ph.D. Qualifying Examination 2024 August (Analysis)

A closed book paper with helpsheet. Do all the questions. Theorems/results used should be clearly stated

Question 1 [12 marks]

Answer only true or false for each of the following statements.

(a) A function on $\mathbb{R}^n$ is (Lebesgue) measurable if and only if it is equal to a continuous function almost everywhere.

(b) If f and g are Riemann integrable on all finite intervals of $\mathbb{R}$ and f has finite support, then their convolution is a continuous function.

(c) Any piecewise differentiable function on $[a, b]$ is a difference of two non-decreasing functions.

A function f is said to be piecewise differentiable on $[a, b]$ if there is a partition of $[a, b] =: a = x_0 < x_1 < \cdots < x_n = b$ such that f' is uniformly continuous on each subinterval (x_j, x_{j+1}) for all $j = 0, \cdots, n-1$.

(d) For any open set $\Omega \subset \mathbb{R}^n$, there exists countable number of pairwise disjoint closed cubes $\{Q_k\}$ in Ω such that $|\Omega \setminus \bigcup Q_k| = 0$.

(e) Any real analytic function on $\mathbb{R}$ can have at most countable number of zeroes.

(f) Let $\{f_k\}$ be a sequence of complex analytic functions on a domain $\Omega \subset \mathbb{C}$ such that it is uniformly bounded on all compact subsets in Ω. Then the sequence has a subsequence that has a limit function that is also complex analytic on Ω.

(g) If $\sum_{k=0}^{\infty} a_k x^k$ converges for all $x > 0$, then it defines a real analytic function on $\mathbb{R}$.

(h) Any function that is of bounded variation has at most a countable number of discontinuities.

Question 2 [6 marks]

Let $\{A_K\}$ be an increasing sequence of sets in $\mathbb{R}^n$. Show that

$$\lim_{k\to\infty} |A_k|_e = |\bigcup_{k=1}^{\infty} A_k|_e$$

where $|\cdot|_e$ denotes the outer measure.

Question 3 [8 marks]

Let H be a Hilbert space. If $\{x_k\}$ is a orthogonal sequence of elements in H. Show that $\sum_{k=1}^{\infty} x_k$ defines an element in H if and only if

$$\sum_{k=1}^{\infty} \|x_k\|^2 < \infty.$$

Question 4 [12 marks]

Given $f > 0$ a.e. is measurable such that both $f, \log f$ are integrable on a measurable set E with measure 1. Show that

$$\lim_{p\to 0^+} \left(\int_E f^p dx \right)^{1/p} = \exp\left(\int_E \log f dx \right).$$

Question 5 [7 marks]

Let $1 < p, q < \infty$ and $f, g : [0, \infty) \to [0, \infty)$ be continuous. Show that

$$\int_0^{\infty} f(t)g(t)dt \le \left(\int_0^{\infty} (t^{1/p} f(t))^q dt/t \right)^{1/q} \left(\int_0^{\infty} (t^{1/p'} f(t))^{q'} dt/t \right)^{1/q'}$$

where $1/p + 1/p' = 1/q + 1/q' = 1$.

Question 6 [8 marks]

Let f be an integrable function (may be complex valued) on $\mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function. Define

$$g(z) = \int_{\mathbb{R}} \frac{f(t)}{\phi(t) - z} dt.$$

Show that the function g is analytic on $\mathbb{C} \setminus \mathbb{R}$.

Question 7 [6 marks]

Let f be a complex valued function that is analytic at $a \in \mathbb{C}$ such that $f'(a) \neq 0$ and $f(a) = 0$. Show that

$$\lim_{r\to 0^+} \frac{e^{-it} f(a + re^{it})}{|f(a + re^{it})|}$$

exists and independent of $t \in \mathbb{R}$.

Question 8 [8 marks]

(a) Find the derivative of $\sqrt{x^2+y^2+z^2}$ whenever it exists.

(b) Let $f(x,y,z) = |\sqrt{x^2+y^2+z^2}|^a$, $a > 0$. Compute

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

at all points where it exists.

Question 9 [8 marks]

Let f be a decreasing nonnegative function on $[0,1]$ and let $g(t) = \int_0^t f(x)dx/t$. Then

$$\int_0^1 g(t)dt = \int_0^1 f(t)\log(1/t)dt.$$

Question 10 [25 marks]

For at most **five** (5) of the following statements, prove or disprove each of the statement considered.

(a) Let f be a nonnegative measurable function on $\mathbb{R}^n$. Then it can be written as

$$\sum_{k=1}^{\infty} a_k 1_{E_k} \quad \text{where } E_k \text{ are measurable and } a_k > 0 \text{ for all } k\ .$$

(b) A (nonempty) perfect set is uncountable. We say a set is perfect if every point in the set is a limit point.

(c) Let $\{E_k\}$ be a sequence of sets. Then

$$\limsup_{k\to\infty} 1_{E_k} = 1_E \quad \text{where } E = \bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty} E_k.$$

(d) Let $f : [a,b) \to \mathbb{R}$. Then f is convex and continuous on $[a,b)$ if and only if there exists a non-decreasing function g on (a,b) such that

$$f(x) = f(a) + \int_a^x g(t)dt \text{ for all } x \in (a,b).$$

(e) Given any infinite subset A of natural numbers, there exists a complex analytic function f such that their k-th derivatives $f^{(k)}(0) = 0$ for all $k \in A$.

(f) Let f be complex analytic on a domain Ω. Then for any $z_0 \in \Omega$, f has a power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

that converges for all $|z - z_0| < R$ where

$$R = \inf\{|z_0 - \xi| : \xi \notin \Omega\}.$$

(g) Let f be a locally integrable function on $\mathbb{R}^n$ and define

$$M_1 f(x) = \sup |Q|^{1-n} \int_Q |f(y)| dy$$

where the supremum is taken over all open cubes that contain x. Then $M_1 f$ is Borel measurable.

(h) Let μ be a signed measure (additive set function) on a measurable set E and f be a measurable function on E. Then

$$\int_E |f(x)| d\mu = \int_0^{\infty} \mu\{x \in E : |f(x)| > t\} dt.$$