Algebra Qualifying Exam
January 2025

Please attempt all problems. Each problem is worth 10 points, except problems 6 and 9 which are each worth 15 points. The total possible score is 100 points.

Problems

1. Classify all groups of order 21.

2. Compute the Galois group of the polynomial $x^4 + 2$ over $\mathbf{Q}$, and draw the lattice of all subfields of its splitting field.

3. Compute the group $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/6\mathbf{Z}, \mathbf{Z}/15\mathbf{Z})$.

4. Precisely state the classification theorem for finitely generated modules over a PID.

5. Prove that if A is a Noetherian ring and M is a finitely generated A-module, then any A-submodule $N \subset M$ is finitely generated.

6. i. Give the precise definition of a Dedekind domain.
   ii. Give the precise definition of an Artinian ring.
   iii. Give the precise statement of the Hilbert basis theorem.

7. Determine the number of irreducible complex representations of the group A_4, along with their dimensions.

8. Classify all abelian groups of order 200 up to isomorphism.

9. Give one example of each of the following:
   i. A commutative ring with exactly three prime ideals.
   ii. A Dedekind domain which is not a principal ideal domain.
   iii. A commutative ring with infinitely many elements but exactly four invertible elements.