PhD QE

NATIONAL UNIVERSITY OF SINGAPORE

Qualifying Exam - Analysis

(January 2025)

Time allowed : 3 hours

<u>INSTRUCTIONS TO CANDIDATES</u>

1. Please write **your name and student number on the cover page** of your solution.
2. This examination paper contains **5** questions and comprises **3** pages (including this cover page).
3. Answer **all** questions and justify your steps.
4. This is a **closed book** examination. No helpsheet is allowed.

Question 1 [30 marks]

Let $\{f_n\}$ be a sequence that is bounded both in $\mathcal{L}^2(\mathbb{R}^d)$ and $\mathcal{L}^\infty(\mathbb{R}^d)$.
Suppose that $f_n \to f$ almost everywhere in $\mathbb{R}^d$ for some $f \in \mathcal{L}^2(\mathbb{R}^d)$.

a) Show that $f_n \to f$ weakly in $\mathcal{L}^2(\mathbb{R}^d)$.

b) If we also assume $\|f_n\|_{\mathcal{L}^2(\mathbb{R}^d)} \to \|f\|_{\mathcal{L}^2(\mathbb{R}^d)}$, show that $f_n \to f$ in $\mathcal{L}^2(\mathbb{R}^d)$.

c) If we do not assume $\|f_n\|_{\mathcal{L}^2(\mathbb{R}^d)} \to \|f\|_{\mathcal{L}^2(\mathbb{R}^d)}$, do we always have $f_n \to f$ in $\mathcal{L}^2(\mathbb{R}^d)$?
If yes, prove the statement. Otherwise, find a counterexample.

Question 2 [10 marks]

Suppose that $f \in \mathcal{L}^1(X, \mu)$, where (X, μ) is a measure space.
Given $\varepsilon > 0$, show that there is $\delta > 0$ such that

$$|\int_A f d\mu| < \varepsilon$$

for all measurable sets A with $\mu(A) < \delta$.

Question 3 [40 marks]

Let (X, d) be a metric space.

For $E \subset X$, we say that E has *Property* T if for any $\varepsilon > 0$, there is a finite collection $\{x_1, x_2, \ldots, x_n\} \subset E$ such that

$$E \subset \cup_{k=1,2,\ldots,n} B_\varepsilon(x_k).$$

Here $B_r(x)$ denotes the open ball with radius r centered at x.

a) If $f : X \to \mathbb{R}$ is uniformly continuous, and $E \subset X$ has Property T, show that $f(E)$ has Property T in $\mathbb{R}$.

b) If $f : X \to \mathbb{R}$ is continuous, and $E \subset X$ has Property T, does $f(E)$ necessarily have Property T? If yes, prove the statement. Otherwise, find a counterexample.

c) Suppose that $K \subset X$ is compact, show that K has Property T.

d) Suppose that $K \subset X$ has Property T and is complete, show that K is compact.

Question 4 [10 marks]

Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function.
If the real part of f is non-negative on $\mathbb{C}$, does f have to be constant?
If yes, prove the statement. Otherwise, find a counterexample.

Question 5 [10 marks]

Let $\{f_n\}$ be a sequence of holomorphic functions on the open unit ball $B_1 \subset \mathbb{C}$.

Under the assumption

$$|f_n(z)| \leq 1 \text{ for all } z \in B_1,$$

show that there is a subsequence $\{f_{n_k}\}$ which is uniformly convergent on any compact subset of B_1.

End of Paper