

Algebra Qualifying Exam

August 2025

Please attempt all problems. Each problem is worth 10 points, except problems 6 and 9 which are each worth 15 points. The total possible score is 100 points.

Problems

1. Classify all groups of order 14.
2. Give an example of a Galois extension $K/\mathbf{Q}$ with Galois group $\mathbf{Z}/3\mathbf{Z}$.
3. Compute the group $\mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/12\mathbf{Z}, \mathbf{Z}/10\mathbf{Z})$.
4. Precisely state the classification theorem for finitely generated modules over a PID.
5. Prove that if A is a PID and M is a finitely generated free A -module, then any A -submodule $N \subset M$ is finitely generated and free.
6.
 - i. Give the precise definition of a Dedekind domain.
 - ii. Give the precise definition of an integrally closed domain.
 - iii. Give the precise statement of Krull's principal ideal theorem.
7. Determine the number of irreducible complex representations of the group S_4 , along with their dimensions.
8. Give an example of an irreducible monic polynomial $f(x) \in \mathbf{Z}[x]$ which is reducible modulo p for all primes p .
9. Give one example of each of the following:
 - i. A commutative ring of Krull dimension 2 which is not a domain.
 - ii. A Dedekind domain which is not a principal ideal domain.
 - iii. A commutative ring with infinitely many elements but exactly six invertible elements.