

NATIONAL UNIVERSITY OF SINGAPORE

Qualifying Exam - Analysis

(August 2025)

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Please write **your name and student number on the cover page** of your solution.
2. This examination paper contains **3** questions and comprises **3** pages (including this cover page).
3. Answer **all** questions and justify your steps.
4. This is a **closed book** examination. No helpsheet is allowed.

Question 1 [30 marks]

For a continuous function $f : [0, 1] \rightarrow \mathbb{R}$, we say that it is $\frac{1}{2}$ -Hölder continuous if $\sup_{x,y \in [0,1], x \neq y} \frac{|f(x)-f(y)|}{|x-y|^{\frac{1}{2}}}$ is finite. In this case, its $\frac{1}{2}$ -Hölder norm is defined as

$$\|f\|_{\frac{1}{2}} := \sup_{x \in [0,1]} |f(x)| + \sup_{x,y \in [0,1], x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\frac{1}{2}}}. \quad (1)$$

The space of $\frac{1}{2}$ -Hölder continuous functions on $[0, 1]$ is denoted by $C^{1/2}([0, 1])$.

- a) Show that $C^{1/2}([0, 1])$ is a normed linear space with the norm in (1).
- b) Let $(f_n) \subset C^{1/2}([0, 1])$ be a sequence satisfying

$$\|f_n\|_{\frac{1}{2}} \leq M \text{ for all } n,$$

where M is a positive constant.

Show that (f_n) contains a subsequence (f_{n_k}) that converges uniformly to some $f \in C^{1/2}([0, 1])$.

- c) Show that $C^{1/2}([0, 1])$ is a Banach space.

Question 2 [30 marks]

We endow the Euclidean space $\mathbb{R}^d$ with the standard Lebesgue measure. For $p \in [1, +\infty)$ and a measurable set $E \subset \mathbb{R}^d$, the space $L^p(E)$ is defined as

$$L^p(E) := \{f : E \rightarrow \mathbb{R} \mid \int_E |f|^p < +\infty\}$$

with the norm

$$\|f\|_{L^p(E)} := \left(\int_E |f|^p \right)^{\frac{1}{p}}.$$

For this question, you can use that $L^p(E)$ is a Banach space with this norm.

- a) For a compact $K \subset \mathbb{R}^d$, show that

$$L^p(K) \subset L^q(K)$$

if $1 \leq q \leq p < +\infty$.

Does the conclusion hold if we remove the compactness assumption on K ? If yes, prove the conclusion. Otherwise, find a counterexample.

- b) For a sequence $(f_n) \subset L^2(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$, we say that f_n converges to f weakly in $L^2(\mathbb{R}^d)$ if

$$\int f_n \varphi \rightarrow \int f \varphi$$

for all $\varphi \in L^2(\mathbb{R}^d)$.

If f_n converges weakly to f in $L^2(\mathbb{R}^d)$, show that (f_n) is bounded in $L^2(\mathbb{R}^d)$.

c) Suppose that f_n converges weakly to f in $L^2(\mathbb{R}^d)$.

If this sequence satisfies

$$\int_{\mathbb{R}^d} |f_n|^2 \rightarrow \int_{\mathbb{R}^d} |f|^2,$$

show that $f_n \rightarrow f$ in $L^2(\mathbb{R}^d)$.

Question 3 [40 marks]

For this problem, let (X, d) be a non-empty metric space.

a) For a point $p \in X$ and a non-empty subset $E \subset X$, define the distance from p to E as

$$D(p, E) := \inf_{e \in E} d(p, e).$$

Suppose that E is closed, show that $p \notin E$ if and only if $D(p, E) > 0$.

b) For two non-empty subsets $A, B \subset X$, define the distance between them as

$$L(A, B) := \max\left\{ \sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A) \right\}.$$

Show that

$$L(A, B) = 0$$

if and only if A and B have the same closure in X .

c) For $A \subset X$ and $r > 0$, define

$$N_r(A) := \{x \in X : D(x, A) < r\}.$$

Show that, for non-empty A and B , we have

$$L(A, B) = \inf\{r > 0 : A \subset N_r(B), \text{ and } B \subset N_r(A)\}.$$

d) Let $C(X)$ denote the space of non-empty compact subsets of X .

Show that $C(X)$ is a metric space with metric L .