

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 2025-2026

Ph.D. QUALIFYING EXAMINATION

PAPER 1

ALGEBRA

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. Marks for each question are indicated at the beginning of the question.
3. This is a closed book examination. No material is allowed to be brought in.

Answer **all** questions.

Question 1 [20 marks]

- (a) Let G be a finite simple group and let H be a proper subgroup of G of index k . Prove that if $|G| > 2$, then G is isomorphic to a subgroup of the alternating group A_k . [10 marks]
- (b) Prove that a group of order $180 (= 2^2 \times 3^2 \times 5)$ can never be simple. [10 marks]

Question 2 [20 marks]

Let R be a non-commutative ring with 1, and suppose that R has a unique maximal left ideal I .

- (a) Show that if $a \in R \setminus I$, then $a \notin Ia$. [10 marks]
(Hint: $\text{Ann}_R(a) := \{r \in R \mid ra = 0\}$ is a proper left ideal of R .)
- (b) Hence, or otherwise, show that:
- (i) I is a two-sided ideal. [4 marks]
- (ii) for all $r \in R$, r is a unit if and only if $r \notin I$. [6 marks]

(You may **NOT** assume any standard result about the Jacobson ideal without proof.)

Question 3 [20 marks]

Let R be a non-commutative ring with 1. Let M be an (R, R) -bimodule and V be a left R -module. Let $\text{Hom}_R(M, V)$ be the set of left R -module homomorphisms from M to V .

Prove or disprove each of the following statements:

- (a) $\text{Hom}_R(M, V)$ is a left R -module if we define $r \cdot \phi$ by $(r \cdot \phi)(m) = r(\phi(m))$ for $r \in R$ and $\phi \in \text{Hom}_R(M, V)$. [5 marks]
- (b) $\text{Hom}_R(M, V)$ is a left R -module if we define $r \cdot \phi$ by $(r \cdot \phi)(m) = \phi(mr)$ for $r \in R$ and $\phi \in \text{Hom}_R(M, V)$. [5 marks]
- (c) $\text{Hom}_R(M, V)$ is a right R -module if we define $\phi \cdot r$ by $(\phi \cdot r)(m) = \phi(mr)$ for $r \in R$ and $\phi \in \text{Hom}_R(M, V)$. [5 marks]
- (d) $\text{Hom}_R(M, V)$ is a right R -module if we define $\phi \cdot r$ by $(\phi \cdot r)(m) = r(\phi(m))$ for $r \in R$ and $\phi \in \text{Hom}_R(M, V)$. [5 marks]

Question 4 [20 marks]

Let L be a finite Galois extension of a field K , with $[L : K] = n$. Let $\text{Gal}(L/K) = \{\sigma_1, \dots, \sigma_n\}$, and let $\{\alpha_1, \dots, \alpha_n\}$ be a basis for L as a vector space over K . Define \mathbf{D} to be the $(n \times n)$ -matrix over L whose (i, j) -entry equals $\sigma_i(\alpha_j)$.

- (a) Show that $\det(\mathbf{D}) \neq 0$. [10 marks]
 (Hint: show that the rows of \mathbf{D} are linearly independent.)
- (b) Show further that $(\det(\mathbf{D}))^2 \in K$. [10 marks]

Question 5 [20 marks]

Let V be an n -dimensional vector space over a field F , and let $\langle -, - \rangle : V \times V \rightarrow F$ be a symmetric bilinear form on V . Suppose that $\langle v, v \rangle \neq 0$ for all nonzero $v \in V$.

- (a) Prove that if $|F| = 3$, then $n \leq 2$ and there exists an orthogonal basis $\{v_1, \dots, v_n\}$ for V such that $\langle v_i, v_i \rangle = \langle v_j, v_j \rangle$ for all $i, j \in \{1, \dots, n\}$. [10 marks]
- (b) Prove that if $|F| = 5$, then $n \leq 2$. [10 marks]

(You may **NOT** assume Hilbert's 90 Theorem or any of its corollaries without proof.)