PhD QE

NATIONAL UNIVERSITY OF SINGAPORE

Qualifying Exam - Analysis

(January 2026)

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Please write **your name and student number on the cover page** of your solution.
2. This examination paper contains **3** questions and comprises **4** pages (including this cover page).
3. Answer **all** questions and justify your steps.
4. This is a **closed book** examination. No helpsheet is allowed.

Conventions and Notations

- Functions in this exam are defined on (subsets of) the Euclidean space $\mathbb{R}^d$ and are real-valued.
- The Euclidean space $\mathbb{R}^d$ is endowed with the Lebesgue measure. The Lebesgue measure of a set E is denoted by $|E|$.
- The ball with radius r and center x is denoted by $B_r(x)$.

Question 1 [40 marks]

A function f is *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $t \in [0,1]$ and $x, y \in \mathbb{R}^d$.

Let $\{f_n\}$ denote a sequence of functions on $\mathbb{R}^d$.

We say that this sequence is *locally uniformly bounded* if for each $R > 0$, there is a constant C_R such that

$$\sup_{B_R(0)} |f_n| \leq C_R \quad \text{for all } n.$$

We say that f_n converges to f *locally uniformly* if for any compact set K, we have $f_n \to f$ uniformly on K.

Answer the following questions.

a) Suppose that f is differentiable and convex on $\mathbb{R}^d$.
Show that f attains its minimum at 0 if and only if $\nabla f(0) = 0$.

b) Suppose that $\{f_n\}$ is a locally uniformly bounded sequence of convex functions on $\mathbb{R}^d$.
Show that there is a convex function f and a subsequence $\{f_{n_k}\}$ such that f_{n_k} converges to f locally uniformly.

c) Let $\{f_n\}$ be a sequence of convex functions on $\mathbb{R}^d$ that converges to f locally uniformly.
Suppose that f_n attains its minimum value in $\mathbb{R}^d$ at p_n, and that $p_n \to p \in \mathbb{R}^d$, does f take its minimum value at p?
Justify your answer.

d) Suppose that $\{f_n\}$ is a sequence of convex functions on $\mathbb{R}^d$ that converges to f locally uniformly.
If f_n attains its minimum at $p_n \in \mathbb{R}^d$ and f attains its minimum at p, do we always have $p_n \to p$?
Justify your answer.

Question 2 [30 marks]

For a measurable set E in $\mathbb{R}^d$, define its *density function* as

$$\delta_E(x) := \lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|}.$$

With this, the *measure theoretic boundary* of E is defined as

$$\partial_M E := \{x \in \mathbb{R}^d : \ \delta_E(x) \text{ is not defined, or } 0 < \delta_E(x) < 1\}.$$

a) Show that $\partial_M E$ has measure 0.

b) For the unit ball B_1, show that

$$\partial_M B_1 = \partial B_1,$$

where ∂B_1 is the topological boundary of B_1.

c) For a measurable set E in $\mathbb{R}^d$, do we always have

$$\partial_M E = \partial E?$$

Justify your answer.

Question 3 [30 marks]

Recall that $C^1([0,1])$ denotes the space of continuously differentiable functions on $[0,1]$. Define a subset as

$$K := \{f \in C^1([0,1]) : \ f(0) = 0, \ f(1) = 1\}.$$

a) Define a functional on K as

$$\Phi(f) = \int_0^1 (f(x))^2 dx.$$

What is the infimum of Φ over K? Is this infimum attained by a function $f \in K$?

Justify your answer.

b) Define a functional on K as

$$\Psi(f) = \int_0^1 (f'(x))^2 dx.$$

What is the infimum of Ψ over K? Is this infimum attained by a function $f \in K$?

Justify your answer.

End of Paper