NATIONAL UNIVERSITY OF SINGAPORE, DEPARTMENT OF MATHEMATICS

Ph.D. Qualifying Examination
Year 2025-2026 Semester II
Computational Mathematics

Time allowed : 3 hours

Instructions to Candidates

1. Use A4 size paper and pen (blue or black ink) to write your answers.
2. Write down your student number clearly on the top left of every page of the answers.
3. Write on one side of the paper only. Start each question on a NEW page. Write the question number and page number on the top right corner of each page (e.g. Q1P1, Q1P2, $\cdots$, Q2P1, $\cdots$).
4. This examination paper comprises two parts: Part I contains THREE (3) questions and Part II contains THREE (3) questions. Answer ALL questions.
5. The total mark for this paper is ONE HUNDRED (100).
6. This is a CLOSED BOOK examination: you are allowed to bring a help sheet.
7. You may use any calculator. However, you should lay out systematically the various steps in the calculations

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Part I: Scientific Computing

Question 1 [25 marks]

Let A be non-singular. Show that

$$\min\{\frac{\|\delta A\|_2}{\|A\|_2} : \; A + \delta A \text{ is singular}\} = \frac{1}{k_2(A)},$$

where $k_2(A)$ denotes the 2-norm condition number of A.

Question 2 [20 marks]

When the problem

$$\begin{aligned} u_t + 2u_x &= 0, \; -\infty < x < \infty, t > 0, \\ u &= 10^{x^2}, \; -\infty < x < \infty, \; t = 0 \end{aligned}$$

is solved using the Backward and the Forward finite difference scheme with $h = 0.0001$ and $\tau = 0.00001$, respectively, at the grid point $(0, \tau)$, we obtain the approximations $u = 1.000000004$ and $u = 0.999999995$, respectively. Explain which of these two values is more accurate.

Question 3 [20 marks]

The equation

$$r\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - f(x,t) = 0,$$

with r being constant is approximated at the point $(ih, j\tau)$ in the $x-t$ plane by the difference scheme

$$\frac{r}{\alpha}\{u_{i,j+1} - 0.5(u_{i+1,j} + u_{i-1,j})\} + 0.5(u_{i+1,j} - u_{i-1,j}) - hf_{i,j} = 0.$$

Investigate the consistency of this scheme, where, α is a positive constant, the solution u of the PDE is sufficient smooth.

Part II: Optimization

Question 1 [8 marks]

Let $f(x) = \max\{x^2, 2x\}$.

1. Compute $\partial f(\bar{x})$, where $\bar{x} = 2$.
2. Compute the Moreau envelope of f.

Question 2 [12 marks]

Consider the following least square problem with an ℓ_1 regularizer

$$\min_{x \in \mathbf{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \mu\|x\|_1, \tag{1}$$

where $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$ are design matrix and target vector, respectively, and $\mu \geq 0$ is a parameter that controls sparsity. It is evident that (1) is equivalent to

$$\begin{aligned} \min_{x,r} \ & \frac{1}{2}\|r\|_2^2 + \mu\|x\|_1 \\ \text{s.t.} \ & Ax - b = r. \end{aligned} \tag{2}$$

1. Write down the KKT conditions for (2).
2. Show that the dual problem of (2) can be written as

$$\begin{aligned} \max_{\lambda} \ & b^\top \lambda - \frac{1}{2}\|\lambda\|_2^2 \\ \text{s.t.} \ & \|A^\top \lambda\|_\infty \leq \mu. \end{aligned}$$

Question 3 [15 marks]

Consider the following equality constrained optimization problem

$$\begin{aligned} \min_{x} \ & f(x) \\ \text{s.t.} \ & c_i(x) = 0 \quad \forall i = 1, 2, \ldots, m, \end{aligned} \tag{3}$$

where f and all c_i $(i = 1, \ldots, m)$ are finite continuous functions. Assume that an optimal solution x^* exists. Let M be an estimate of $f(x^*)$. Define the nonlinear least square function

$$F(x, M) = [f(x) - M]^2 + \sum_{i=1}^{m} c_i^2(x).$$

Given an initial value $M_0 \leq f(x^*)$, the *Morrison method* generates a sequence $\{(x^k, M_k)\}$ by:

- Compute a global minimizer (assume to exist)

$$x^k \in \operatorname*{argmin}_x F(x, M_k).$$

- Update

$$M_{k+1} = M_k + \sqrt{F(x^k, M_k)}.$$

Prove the following results.

1. Show that if $M_k \leq f(x^*)$, then $f(x^k) \leq f(x^*)$.
2. Show that if $M_k \leq f(x^*)$, then $M_{k+1} \leq f(x^*)$.
3. Assume $\bar{x}$ is any limit of the sequence $\{x^k\}$. Prove that $\bar{x}$ is an optimal solution to (3).